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On Generating Tridiagonal Matrices of Generalized (s, t)-Pell, (s, t)-Pell Lucas and (s, t)-Modified Pell Sequences

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Abstract

In this study, we define some tridigional matrices depending on two real parameters. By using the determinant of these matrices, the elements of (s, t)-Pell, (s, t)-Pell Lucas and (s, t)-modified Pell sequences with even or odd indices are generated. Then we construct the inverse matrices of these tridigional matrices. We also investigate eigenvalues of these matrices.

1 Introduction and Preliminaries

Special integer sequences are encountered in different branches of science, art, nature, the structure of our body. So it is a popular topic in applied mathematics. Two of the special integer sequences are the Pell and Pell Lucas sequences. By changing the initial conditions but preserving the recurrence relation the Pell Lucas sequence is obtained. The authors investigated some sum formulas for Pell numbers in [2]. Gulec and Taskara generalized the Pell and Pell Lucas numbers by using two parameters in [3]. The authors generalized the modified Pell sequence similarly in [4]. By the determinant of the tridiagonal matrix, the values of the Fibonacci and Lucas numbers are demonstrated in [8]. Feng gave Fibonacci identities via determinant of the special tridiagonal

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matrix in [9]. Seibert and Trojovsk studied the factorization of the Fibonacci and Lucas numbers using determinants of tridiagonal matrices in [10]. In [11], it is demonstrated that the permanents of some special tridiagonal matrices are equal to Fibonacci numbers. Falcon displayed some equalities of k-Fibonacci numbers with the determinant of some special matrices in [12]. Catarino obtained the n-th elements of k-Pell, k-Pell-Lucas, and modified k-Pell sequences by the determinants of some tridiagonal matrices in [13]. In [14], properties of hyperbolic generalized Pell numbers are studied. The recurrence relations for Pell, Pell Lucas and modified Pell sequences are $P_n = 2P_{n-1} + P_{n-2},\ P_0 = 0,\ P_1 = 1$, $Q_n = 2Q_{n-1} + Q_{n-2},\ \ Q_0 = 2,\ Q_1 = 2$ and $q_n = 2q_{n-1} + q_{n-2},\ \ q_0 = 1,\ q_1 = 1$ for $n \geq 2$, respectively in [1]. There are some generalizations of these integer sequences. For example, the generalizations for Pell, Pell Lucas, and modified Pell sequences called (s,t)-Pell, (s,t)-Pell Lucas, (s,t)-modified Pell sequences are defined by the aid of the following recurrence relations respectively

$$\begin{aligned}
\wp_n(s,t) &= 2s\wp_{n-1}(s,t) + t\wp_{n-2}(s,t), \quad \wp_0(s,t) = 0, \wp_1(s,t) = 1, \\
\Re_n(s,t) &= 2s\Re_{n-1}(s,t) + t\Re_{n-2}(s,t), \quad \Re_0(s,t) = 2,\ \Re_1(s,t) = 2s, \\
\aleph_n(s,t) &= 2s\aleph_{n-1}(s,t) + t\aleph_{n-2}(s,t), \quad \aleph_0(s,t) = 1,\ \aleph_1(s,t) = s,
\end{aligned}$$

for $n \geq 2$ in [3, 4].

Some elements of (s,t)-Pell and (s,t)-Pell Lucas sequences are given in the following tables:

n	$(s,t)-$ Pell numbers
1	1
2	$2s$
3	$4s^2 + t$
4	$8s^3 + 4st$
5	$16s^4 + 12s^2t + t^2$
6	$32s^5 + 32s^3t + 6st^2$
7	$64s^6 + 80s^4t + 24s^2t^2 + t^3$

n	$(s,t)-$ Pell Lucas numbers
1	$2s$
2	$4s^2+2t$
3	$8s^3+6st$
4	$16s^4+16s^2t+2t^2$
5	$32s^5+40s^3t+10st^2$
6	$64s^6+96s^4t+36s^2t^2+2t^3$
7	$128s^7+224s^5t+112s^3t^2+14st^3$

The elements of (s,t)-modified Pell sequences are obtained by taking half of the (s,t)-Pell Lucas numbers.

Special integer sequences are obtained with special numerical choices for (s,t)-Pell and (s,t)-Pell Lucas, (s,t)-modified Pell sequences. For example, if $s=t=1$, then we get well-known Pell, Pell Lucas, modified Pell sequences. If $s=1$, $t=k$, then we get k-Pell, k-Pell Lucas, and modified k-Pell sequences.

Let us consider a tridiagonal matrix as

$$A_n = \begin{bmatrix} a & b & & & & & \\ c & d & e & & & & \\ & c & d & e & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & & c & d & e \\ & & & & & c & d \end{bmatrix}.$$

Then

$$\det A_n = d\det A_{n-1} - ce\det A_{n-2}. \tag{1}$$

The inverse of a matrix A can be obtained by the formula $A^{-1}=\frac{(cof(A))^T}{\det A}$, where $(cof(A))^T$ is the transpose of the cofactor matrix A or adjugate matrix of A [4]. Let T be a nonsingular tridiagonal matrix as

$$T=\begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix}.$$

Usmani [6] gave a formula for the inverse of this matrix $T^{-1}=(t_{i,j})$ as

$$t_{i,j}=\begin{cases} (-1)^{i+j}\frac{1}{\theta_n}b_i...b_{j-1}\theta_{i-1}\phi_{j+1} & \text{if } i\leq j \\ (-1)^{i+j}\frac{1}{\theta_n}c_j...c_{i-1}\theta_{j-1}\phi_{i+1} & \text{if } i>j \end{cases} \tag{2}$$

where

• θ_i verify the recurrence relation $\theta_i=a_i\theta_{i-1}-b_{i-1}c_{i-1}\theta_{i-2}$ for $i=2,....n$, with the initial conditions $\theta_0=1$ and $\theta_1=a_1$. Observe that $\theta_n=\det(T)$.

• ϕ_i verify the recurrence relation $\phi_i=a_i\phi_{i+1}-b_ic_i\phi_{i+2}$ for $i=n-1,...1$, with the initial conditions $\phi_{n+1}=1$ and $\phi_n=a_n$.

In [7], if the tridiagonal matrix is given in the following form

$$T=\begin{bmatrix} a & b & & & \\ c & a & b & & \\ & c & a & \ddots & \\ & & \ddots & \ddots & b \\ & & & c & a \end{bmatrix},$$

then the eigenvalues of this matrix are

$$\lambda_r=a+2\sqrt{bc}\cos(\frac{r\pi}{n+1}), \qquad r=1,2,...,n. \tag{3}$$

1.1 Some properties of tridiagonal matrices $A_{p,n}$ by (s,t)-Pell sequence

Theorem 1. *Assume that $A_{p,n}$ is an $n\times n$ tridiagonal matrix defined as*

$$A_{p,n}=\begin{bmatrix} 2s & t & & & & \\ -1 & 2s & t & & & \\ & -1 & 2s & & & \\ & & & & \ddots & \\ & & & \ddots & \ddots & t \\ & & & & -1 & 2s \end{bmatrix}.$$

Then the determinant of $A_{p,n}$ is

$$\det(A_{p,n})=\wp_{n+1}.$$

Proof. The proof is made by induction applied on n. For $n=1$, we have $\det(A_{p,1})=\wp_2=2s$. Assume that $\det(A_{p,n-1})=\wp_n$, $\det(A_{p,n})=\wp_{n+1}$ for $n>2$. Then

$$\begin{aligned}\det(A_{p,n+1}) &= 2s\det(A_{p,n})-(-t)\det(A_{p,n-1})\\ &= 2s\wp_{n+1}+t\wp_n=\wp_{n+2}.\end{aligned}$$

■

(s,t)-Pell sequence is also obtained by using the following tridiagonal matrix with complex entries. Assume that A_n is an $n\times n$ matrix defined as

$$A_n=\begin{bmatrix} 2s & it & & & \\ i & 2s & it & & \\ & i & 2s & \ddots & \\ & & \ddots & \ddots & \\ & & & & it \\ & & & i & 2s \end{bmatrix}.$$

Then it is easily seen that the determinant of A_n is also $(n+1)$th element of the (s,t)-Pell sequence

$$\det(A_n) = \wp_{n+1}.$$

For the inverse of $A_{p,n}$, by using (2), it is obtained that

$$\begin{aligned} a_i &= 2s, \quad b_i = t, \quad c_i = -1, \\ \theta_i &= \wp_{i+1}, \quad \phi_j = \wp_{n-j+2}. \end{aligned}$$

Therefore the inverse of $A_{p,n}$ is displayed by

$$(A_{p,n}^{-1})_{(i,j)} = \begin{cases} (-1)^{i+j} t^{j-i} \wp_i \wp_{n-j+1} \frac{1}{\wp_{n+1}}, & \text{if} \quad i \leq j \\ \wp_j \wp_{n-i+1} \frac{1}{\wp_{n+1}}, & \text{if} \quad i > j \end{cases}.$$

The elements of the cofactor matrix are given as

$$cof(A_{p,n})_{(i,j)} = \begin{cases} \wp_i \wp_{n-j+1}, & \text{if} \quad i < j \\ (-1)^{i+j} t^{i-j} \wp_j \wp_{n-i+1}, & \text{if} \quad i \geq j \end{cases}.$$

It is well-known that $|cof(A_{p,n})| = |adj(A_{p,n})| = |A_{p,n}|^{n-1} = \wp_{n+1}^{n-1}$. By using cofactor matrix, we get some properties of (s,t)-Pell sequence. For $n = 2$, we get

$$|cof(A_{p,2})| = \begin{vmatrix} \wp_1 \wp_2 & \wp_1 \wp_1 \\ -t\wp_1 \wp_1 & \wp_2 \wp_1 \end{vmatrix}$$

$\wp_2^2 + t\wp_1^2 = \wp_3$. For $n = 3$, we get

$$|cof(A_{p,3})| = \begin{vmatrix} \wp_1 \wp_3 & \wp_1 \wp_2 & \wp_1 \wp_1 \\ -t\wp_1 \wp_2 & \wp_2 \wp_2 & \wp_2 \wp_1 \\ t^2 \wp_1 \wp_1 & -t\wp_2 \wp_1 & \wp_3 \wp_1 \end{vmatrix}$$

$$\begin{aligned} |cof(A_{p,3})| &= \wp_1 \wp_3 (\wp_2^2 \wp_3 \wp_1 + t\wp_2^2 \wp_1^2) \\ &\quad + t\wp_1 \wp_2 (\wp_1^2 \wp_3 \wp_2 + t\wp_1^3 \wp_2) + t^2 \wp_1^2 (\wp_1^2 \wp_2^2 - \wp_1^2 \wp_2^2) \\ &= \wp_1^2 \wp_2^2 \wp_3 (\wp_3 + t\wp_1) + t\wp_1^3 \wp_2^2 (\wp_3 + t\wp_1) \\ &= (\wp_3 + t\wp_1)^2 \wp_1^2 \wp_2^2 \\ &= \wp_1^2 (\wp_3 + t\wp_1)^2 (\frac{\wp_3 - t\wp_1}{2s})^2 \\ \wp_4^2 &= (\frac{\wp_3^2 - t^2 \wp_1^2}{s})^2. \end{aligned}$$

For $n = 4$, we get

$$|cof(A_{p,4})| = \begin{vmatrix} \wp_1\wp_4 & \wp_1\wp_3 & \wp_1\wp_2 & \wp_1\wp_1 \\ -t\wp_1\wp_3 & \wp_2\wp_3 & \wp_2\wp_2 & \wp_2\wp_1 \\ t^2\wp_1\wp_2 & -t\wp_2\wp_2 & \wp_3\wp_2 & \wp_3\wp_1 \\ -t^3\wp_1\wp_1 & t^2\wp_2\wp_1 & -t\wp_3\wp_1 & \wp_4\wp_1 \end{vmatrix}$$

$$\wp_5^3 = (t\wp_2^2 + \wp_3^2)(t\wp_1\wp_3 + \wp_2\wp_4)^2.$$

Eigenvalues of the matrices $A_{p,n}$ construct the spectra of the $A_{p,n}$. By using the property (3), the sequence of the spectra of $A_{p,n}$ for $n = 1, 2, 3, 5$ is

$n = 1 \Rightarrow \lambda_r = \{2s\}$

$n = 2 \Rightarrow \lambda_r = \{2s + i\sqrt{t},\ 2s - i\sqrt{t}\}$

$n = 3 \Rightarrow \lambda_r = \{2s + i\sqrt{2t},\ 2s - i\sqrt{2t}, 2s\}$

$n = 5 \Rightarrow \lambda_r = \{2s + i2\sqrt{t},\ 2s + i\sqrt{3t}, 2s,\ 2s - i2\sqrt{t},\ 2s - i\sqrt{3t}\}.$

If $s = t = 1$, the sequence of the spectra of the matrices $A_{p,n}$ for $n = 2, 3, 4, 5, 6$ is computed by using the Matlab Program as

$S_2 = \{2 + i,\ 2 - i\}$,

$S_3 = \left\{2 + \sqrt{2}i,\ 2,\ 2 - \sqrt{2}i\right\}$,

$$S_4 = \left\{ \begin{array}{c} 2 + 1.618033988749896i,\ 2 + 0.618033988749895i, \\ 2 - 0.618033988749895i,\ 2 - 1.618033988749896i \end{array} \right\},$$

$$S_5 = \left\{ \begin{array}{c} 2 + 1.732050807568879i,\ 2 - 1.732050807568879i \\ 2,\ 2 + i,\ 2 - i \end{array} \right\},$$

$$S_6 = \left\{ \begin{array}{c} 2 + 1.801937735804838i,\ 2 - 1.801937735804838i, \\ 2 + 0.445041867912629i,\ 2 - 0.445041867912629i, \\ 2 + 1.246979603717468i,\ 2 - 1.246979603717468i \end{array} \right\}.$$

Evidently, the product of eigenvalues is the determinant of the matrix and the sum of eigenvalues is the trace of the matrix. Therefore

$$\sum_{i=1}^{n} \lambda_i = tr(A_{p,n}) = 2sn$$

and

$$\prod \lambda_i = \det(A_{p,n}) = \wp_{n+1} = \prod_{j=1}^{n} (2s + \sqrt{2}i\cos(\frac{\pi j}{n+1})).$$

1.2 Some properties of tridiagonal matrices $E_{p,n}$ by even (s,t)-Pell sequence

Assume that $E_{p,n}(p)$ is an $n \times n$ tridiagonal matrix defined as

$$E_{p,n} = \begin{bmatrix} 2s & 0 & & & & \\ t & 4s^2+2t & t & & & \\ & t & 4s^2+2t & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & t \\ & & & & t & 4s^2+2t \end{bmatrix}.$$

Then the determinant of $E_{p,n}$ is computed by (1) as

$$\det E_{p,n} = \wp_{2n}.$$

For the inverse of $E_{p,n}$, the values in (2) are computed as

$$\begin{aligned} a_1 &= 2s, \\ a_i &= 4s^2+2t, \quad i \geq 2 \\ b_i &= t, \quad i > 1 \\ b_1 &= 0, \\ c_i &= t, \\ \theta_i &= \wp_{2i}, \quad i \geq 1 \\ \phi_j &= \frac{1}{2s}\wp_{2(n-j+2)}, \quad j \geq 1. \end{aligned}$$

Therefore the inverse of $E_{p,n}$ is displayed as

$$(E_{p,n}^{-1})_{(i,j)} = \begin{cases} 0, & \text{if } \quad i=1 \\ (-1)^{i+j} t^{j-i} \wp_{2i-2}\wp_{2(n-j+1)} \frac{1}{2s\wp_{2n}}, & \text{if } \quad i \leq j \\ (-1)^{i+j} t^{i-j} \wp_{2j-2}\wp_{2(n-i+1)} \frac{1}{2s\wp_{2n}}, & \text{if } \quad i > j \end{cases}.$$

If all entries of the matrix are real and nonnegative, then the matrix is called positive. All eigenvalues are real if the matrix positive and tridiagonal [4].

Therefore all eigenvalues of $E_{p,n}$ are real if $s,t \geq 0$. If we choose $s = t = 1$, then the sequence of the spectra of the matrix $E_{p,n}$ for $n = 2,3,4,5,6$ is given in the following result with the help of the Matlab program

$S_2 = \{2, 6\}$

$S_3 = \{2,\ 5,\ 7\}$

$S_4 = \{2,\ 4.5857864376269042,\ 6,\ 7.414213562373092\}$

$S_5 = \left\{ \begin{array}{c} 2, 4.3819660112501022,\ 5.381966011250107, \\ 6.618033988749892,\ 7.618033988749891 \end{array} \right\}$

$S_6 = \{2,\ 4.267949192431121,\ 5,\ 6,\ 7,\ 7.732050807568871\}$.

Evidently

$\sum \lambda_i = tr(E_{p,n}) = (n-1)(4s^2 + 2t) + 2s$ and $\prod \lambda_i = \det(E_{p,n}) = \wp_{2n}$.

If we take care of the spectra, one of the eigenvalues is $2 = 2s$ if $s = t = 1$ for all positive integer n. And the minimum eigenvalue of spectra converges to $2 = 2s$, the maximum eigenvalue of spectra converges to $4s^2 + 4t$.

1.3 The properties of tridiagonal matrices $O_{p,n}$ by odd (s,t)-Pell sequence

Assume that $O_{p,n}$ is an $n \times n$ tridiagonal matrix defined as

$$O_{p,n} = \begin{bmatrix} 4s^2+t & t & & & \\ t & 4s^2+2t & t & & \\ & t & 4s^2+2t & \ddots & \\ & & \ddots & \ddots & t \\ & & & t & 4s^2+2t \end{bmatrix}.$$

Then the determinant of $O_{p,n}$ is given by (1) as

$$\det O_{p,n} = \wp_{2n+1}.$$

For the inverse of $O_{p,n}$, the values are computed by (2) as

$$\begin{aligned} a_1 &= 4s^2+t, \\ a_i &= 4s^2+2t, \quad i \geq 2 \\ b_i &= c_i = t, \quad i \geq 1 \\ \theta_i &= \wp_{2i+1}, \quad i \geq 1 \\ \phi_j &= \frac{1}{2s}\wp_{2(n-j+2)}, \quad j \geq 1. \end{aligned}$$

Therefore the inverse of $O_{p,n}$ is given by

$$(O_{p,n}^{-1})_{(i,j)} = \begin{cases} (-1)^{i+j} t^{j-i} \wp_{2i-1}\wp_{2(n-j+1)} \frac{1}{2s\wp_{2n+1}}, & \text{if} \quad i \leq j \\ (-1)^{i+j} t^{i-j} \wp_{2j-1}\wp_{2(n-i+1)} \frac{1}{2s\wp_{2n+1}}, & \text{if} \quad i > j \end{cases}.$$

Matrices $O_{p,n}$ are symmetric so the eigenvalues are real. If $s=t=1$, the sequence of the spectra of the matrices $O_{p,n}$ for $n=2,3,4,5,6$ is given in the following:

$S_2 = \{4.381966011250105,\ 6.618033988749895\}$

$S_3 = \{4.198062264195162,\ 5.554958132087372,\ 7.246979603717467\}$

$S_4 = \{4.120614758428182,\ 5,\ 6.347296355333861,\ 7.532088886237956\}$

$$S_5 = \left\{ \begin{array}{c} 4.081014052771005,\ 4.690278532109430,\ 5.715370323453431, \\ 6.830830026003771,\ 7.682507065662362 \end{array} \right\}$$

$$S_6 = \left\{ \begin{array}{c} 4.058116365147897,\ 4.502978503657798,\ 5.290790225914926, \\ 6.241073360510647,\ 7.136129493462313,\ 7.770912051306419 \end{array} \right\}.$$

Then

$\sum \lambda_i = tr(O_{p,n}) = (n-1)(4s^2+2t) + (4s^2+t) = n(4s^2+2t) - t$

$\prod \lambda_i = \det(O_{p,n}) = \wp_{2n+1}.$

If we take care of the spectra, minimum eigenvalue converges to $4s^2$. The maximum eigenvalue of spectra converges to $4s^2+4t$.

Theorem 2. *If λ_i is an eigenvalue of the matrix $O_{p,n}$, then $\lambda_i + 8s + 4$ is an eigenvalue of $O_{p,n}(s+1)$.*

Proof. λ_i is an eigenvalue of $O_{p,n}$, then

$$\begin{aligned}
& |O_{p,n} - \lambda_i I| \\
= & \begin{bmatrix} 4(s+1)^2 + t - (\lambda_i + 8s + 4) & t & & & \\ t & \ddots & & & \\ & \ddots & & & \\ & & & & t \\ & & & t & 4(s+1)^2 + 2t - (\lambda_i + 8s + 4) \end{bmatrix} \\
= & \left| O_{p,n}(s+1) - (\lambda_i + 8s + 4) I \right|.
\end{aligned}$$

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1.4 Some properties of tridiagonal matrices $A_{Q,n}$ by (s,t)-Pell Lucas sequence

Assume that $A_{Q,n}$ is an $n \times n$ tridiagonal matrix defined as

$$A_{Q,n} = \begin{bmatrix} 2s & 2t & & & \\ -1 & 2s & t & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & & t \\ & & & -1 & 2s \end{bmatrix}.$$

Then the determinant of $A_{Q,n}$

$$\det A_{Q,n} = \Re_n.$$

(s,t)-Pell Lucas sequence is also obtained by using the following symmetric matrix with complex entries. Assume that A_n is an $n \times n$ tridiagonal matrix defined as

$$A_n = \begin{bmatrix} 2s & 2it & & & & \\ i & 2s & it & & & \\ & i & 2s & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & it \\ & & & & i & 2s \end{bmatrix}.$$

Then the determinant of A_n is given by (1) as

$$\det A_n = \Re_n.$$

The sequence of the spectra of the matrices $A_{Q,n}$ for $n = 2,3,4,5,6$ is given in the following:

$S_2 = \{2 + 1.414213562373095i,\ 2 - 1.414213562373095i\}$

$S_3 = \{2,\ 2 + 1.732050807568877i,\ 2 + 1.732050807568877i\}$

$$S_4 = \left\{ \begin{array}{c} 2 + 1.847759065022573i,\ 2 - 1.847759065022573i \\ 2 + 0.765366864730179i,\ 2 - 0.765366864730179i \end{array} \right\}$$

$$S_5 = \left\{ \begin{array}{c} 2 + 1.902113032590306i,\ 2 - 1.902113032590306i \\ ,2,\ 2 + 1.175570504584946i,\ 2 - 1.175570504584946i \end{array} \right\}$$

$$S_6 = \left\{ \begin{array}{cc} 2 + 1.931851652578137i, & 2 - 1.931851652578137i, \\ 2 + 1.414213562373096i, & 2 - 1.414213562373096i, \\ 2 + 0.517638090205041i, & 2 - 0.517638090205041i \end{array} \right\}.$$

Evidently, $\sum \lambda_i = tr(A_{Q,n}) = 2sn$ and $\prod \lambda_i = \det(A_{Q,n}) = \Re_n$.

For the inverse of $A_{Q,n}$, by using (2), it is obtained that

$$\begin{aligned} a_i &= 2s, \quad i > 0 \\ b_1 &= 2t, \quad b_i = t, \qquad i > 1 \\ c_i &= -1, \quad i > 0 \\ \theta_0 &= 1, \quad \theta_i = \Re_i, \qquad i > 1 \\ \phi_j &= \wp_{n-j+2}, \quad j > 0. \end{aligned}$$

Therefore the inverse of $A_{Q,n}$ is the following matrix

$$(A_{Q,n}^{-1})_{(i,j)} = \begin{cases} (-1)^{i+j} t^{j-i+1} \Re_{i-1} \wp_{n-j+1}(x) \frac{1}{\Re_n}, & \text{if } \ i \leq j \\ (-1)^{1+j} t^{j} \frac{\wp_{n-j+1}}{\Re_n}, & \text{if } \ i = 1, \ j > 1 \\ \frac{\wp_n}{\Re_n} & \text{if } \ i = j = 1 \\ \frac{\wp_{n-i+1}}{\Re_n}, & \text{if } \ j = 1, \ i > 1 \\ \Re_{j-1} \wp_{n-i+1} \frac{1}{\Re_n}, & \text{if } \ i > j \end{cases}.$$

1.5 Some properties of tridiagonal matrices $E_{Q,n}$ by even (s,t)-Pell Lucas sequence

Assume that $E_{Q,n}$ is a $n \times n$ tridiagonal matrix defined as

$$E_{Q,n} = \begin{bmatrix} 4s^2+2t & 2t & & & \\ t & 4s^2+2t & t & & \\ & t & \ddots & \ddots & \\ & & \ddots & & t \\ & & & t & 4s^2+2t \end{bmatrix}.$$

Then the determinant of $E_{Q,n}$ is given by (1)

$$\det E_{Q,n} = \Re_{2n}.$$

For the inverse of $E_{Q,n}$, the values are computed as,

$$\begin{aligned} a_i &= 4s^2+2t, \quad i > 0 \\ b_1 &= 2t, \ b_i = t, \ i > 1 \\ c_i &= t, \quad i \geq 1 \\ \theta_0 &= 1, \quad \theta_i = \Re_{2i}, \\ \phi_j &= \frac{\wp_{2(n-j+2)}}{2s}, \qquad j \geq 1 \end{aligned}$$

where

$$(E_{Q,n}^{-1}) = \begin{cases} (-1)^{i+j} \frac{1}{2s\Re_{2n}} t^{j-i} \Re_{2i-2} \wp_{2(n-j+1)} & \text{if } \ i \leq j \\ (-1)^{i+j} \frac{1}{2s\Re_{2n}} t^{j-i} \Re_{2j-2} \wp_{2(n-i+1)} & \text{if } \ i > j \end{cases}.$$

The sequence of the spectra of the matrices $E_{Q,n}$ for $n = 2, 3, 4, 5, 6$, are given in the following

$S_2 = \{4.585786437626905,\ 7.414213562373095\}$

$S_3 = \{4.267949192431122, 5.999999999999997, 7.732050807568876\}$

$$S_4 = \left\{ \begin{array}{c} 4.152240934977428,\ 5.234633135269817, \\ 7.847759065022574,\ 6.765366864730179 \end{array} \right\}$$

$$S_5 = \left\{ \begin{array}{c} 4.097886967409691,\ 4.824429495415054,\ 6, \\ 7.175570504584945,\ 7.902113032590311 \end{array} \right\}$$

$$S_6 = \left\{ \begin{array}{c} 4.068148347421865,\ 4.585786437626904, \\ 5.482361909794957,\ 6.517638090205042, \\ 7.414213562373094,\ 7.931851652578145 \end{array} \right\}.$$

Then, $\sum \lambda_i = tr(E_{Q,n}) = n(4s^2 + 2t)$ and $\prod \lambda_i = \det(E_{Q,n}) = \Re_{2n}$.

If we take care of the spectra, minimum eigenvalue converges to $4s^2$. The maximum eigenvalue of spectra converges to $4s^2 + 4t$.

1.6 Some properties of tridiagonal matrices $O_{Q,n}$ by odd (s, t)-Pell Lucas sequence

Assume that $O_{Q,n}$ is an $n \times n$ tridiagonal matrix defined as

$$O_{Q,n} = \begin{bmatrix} 2s & -2s & & & \\ t & 4s^2+2t & t & & \\ & t & 4s^2+2t & \ddots & \\ & & \ddots & \ddots & t \\ & & & t & 4s^2+2t \end{bmatrix},$$

then the determinant of $O_{Q,n}$

$$\det O_{Q,n} = \Re_{2n-1}.$$

For the inverse of $O_{Q,n}$, the values are computed by (2)

$$\begin{aligned}
a_1 &= 2s, \\
a_i &= 4s^2+2t,\ i\geq 2 \\
c_1 &= -2s, \quad b_1=t \\
b_i &= c_i=t, \quad i>1 \\
\theta_0 &= 1, \quad \theta_i=\Re_{2i-1}, \quad i\geq 1 \\
\phi_j &= \frac{\wp_{2(n-j+2)}}{2s}, \quad j\geq 1.
\end{aligned}$$

If $s=t=1$, then the sequence of the spectra of the matrices $O_{Q,n}$ for $n=2,3,4,5,6$ is given in the following

$S_2=\{2.585786437626905,\ 5.414213562373095\}$

$S_3=\{2.657076917222828,\ 4.529316580128842,\ 6.813606502648331\}$

$$S_4=\left\{\begin{array}{c} 2.665585781661023,\ 4.258036215697276, \\ 5.741963784302725,\ 7.334414218338978 \end{array}\right\}$$

$$S_5=\left\{\begin{array}{c} 2.666546286933180,\ 4.149626499380563,\ 5.132659714556408 \\ 6.474094390850837,\ 7.577073108279015 \end{array}\right\}$$

$$S_6=\left\{\begin{array}{c} 2.666653286578264,\ 4.096987803756087, \\ 4.780438656766319,\ 5.834296887585696, \\ 6.913255181694661,\ 7.708368183618974 \end{array}\right\}.$$

The sequence of maximum eigenvalue is increasing and converges to $4s^2+4t=8$. So we can say $\lim\limits_{n\to\infty}\max(\lambda(O_n(k))=4s^2+4t$.

Evidently, $\sum\lambda_i=tr(O_{Q,n})=(n-1)(4s^2+2t)+2s$ and $\prod\lambda_i=\det(O_{Q,n})=\Re_{2n-1}$.

1.7 Some properties of tridiagonal matrices $A_{q,n}$ by (s,t)-Modified Pell sequence

Assume that $A_{q,n}$ is an $n\times n$ tridiagonal matrix defined as

$$A_{q,n} = \begin{bmatrix} s & t & & & \\ -1 & 2s & t & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & & t \\ & & & -1 & 2s \end{bmatrix},$$

then the determinant of $A_{q,n}$ is

$$\det A_{q,n} = \aleph_n.$$

(s,t)-modified Pell sequence is also obtained by using the following symmetric matrix with complex entries. Assume that A_n is an $n \times n$ tridiagonal matrix defined as

$$A_n = \begin{bmatrix} s & it & & & & \\ i & 2s & it & & & \\ & i & 2s & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & it \\ & & & & i & 2s \end{bmatrix}.$$

Then the determinant of A_n is given by (1) as

$$\det A_n = \aleph_n.$$

The sequence of the spectra of the matrices $A_{q,n}$ for $n = 2, 3, 4, 5, 6$ is given in the following:

$S_2 = \{1.5 + 0.866025403784438i,\ 1.5 - 0.866025403784438i\}$

$$S_3 = \left\{ \begin{array}{c} 1.430159709001947,\ 1.784920145499028 + 1.307141278682045i, \\ 1.784920145499028 - 1.307141278682045i \end{array} \right\}$$

$$S_4 = \left\{ \begin{array}{c} 1.895123382259650 + 1.552491820061880i, \\ 1.895123382259650 - 1.552491820061880i, \\ 1.604876617740350 + 0.506843901805983i, \\ 1.604876617740350 - 0.506843901805983i \end{array} \right\}$$

$$S_5 = \left\{ \begin{array}{c} 1.941818622615235 + 1.691279149514195i, \\ 1.941818622615235 - 1.691279149514195i \\ 1.583716458826265, \\ 1.766323147971634 + 0.885556760232166i, \\ 1.766323147971634 - 0.885556760232166i \end{array} \right\}$$

$$S_6 = \left\{ \begin{array}{c} 1.964532008416013 + 1.775303570695105i, \\ 1.964532008416013 - 1.775303570695105i, \\ 1.678391681959849 + 0.359079022670333i, \\ 1.678391681959849 - 0.359079022670333i, \\ 1.857076309624141 + 1.159515656334833i, \\ 1.857076309624141 - 1.159515656334833i \end{array} \right\}.$$

Evidently, $\sum \lambda_i = tr(A_{q,n}) = 2sn - s$ and $\prod \lambda_i = \det(A_{q,n}) = \aleph_n$.

For the inverse of $A_{q,n}$, by using (2), it is obtained that

$$\begin{aligned} a_1 &= s, \quad a_i = 2s, \quad i > 1 \\ b_i &= t, \qquad i > 0 \\ c_i &= -1, \quad i > 0 \\ \theta_0 &= 1, \quad \theta_i = \aleph_i, \qquad i > 1 \\ \phi_j &= \wp_{n-j+2}, \quad j > 0. \end{aligned}$$

Therefore, the inverse of $A_{q,n}$ is the following matrix

$$(A_{q,n}^{-1})_{(i,j)} = \begin{cases} (-1)^{i+j} t^{j-i+1} \aleph_{i-1} \wp_{n-j+1}(x) \frac{1}{\aleph_n}, & \text{if} \quad i \leq j \\ (-1)^{1+j} t^j \frac{\wp_{n-j+1}}{\aleph_n}, & \text{if} \ \ i = 1, \ j > 1 \\ \frac{\wp_n}{\aleph_n} & \text{if} \quad i = j = 1 \\ \frac{\wp_{n-i+1}}{\aleph_n}, & \text{if} \ \ j = 1, \ i > 1 \\ \aleph_{j-1} \wp_{n-i+1} \frac{1}{\aleph_n}, & \text{if} \quad i > j \end{cases}.$$

1.8 Some properties of tridiagonal matrices $E_{q,n}$ by even (s,t)-Modified Pell sequence

Assume that $E_{q,n}$ is an $n \times n$ tridiagonal matrix defined as

$$E_{q,n} = \begin{bmatrix} 2s^2+t & t & & & \\ t & 4s^2+2t & t & & \\ & t & \ddots & \ddots & \\ & & \ddots & & t \\ & & & t & 4s^2+2t \end{bmatrix}.$$

Then the determinant of $E_{q,n}$ is given by (1)

$$\det E_{q,n} = \aleph_{2n}.$$

For the inverse of $E_{q,n}$, the values are computed by (2)

$$\begin{aligned} a_1 &= 2s^2+t,\ a_i = 4s^2+2t, \quad i>1 \\ b_i &= t,\ i>0 \\ c_i &= t, \quad i \geq 1 \\ \theta_0 &= 1, \quad \theta_i = \aleph_{2i}, \\ \phi_j &= \frac{\wp_{2(n-j+2)}}{2s}, \quad j \geq 1. \end{aligned}$$

Therefore, we get

$$(E_{q,n})^{-1}{}_{i,j} = \begin{cases} (-1)^{i+j} \frac{1}{2s\aleph_{2n}} t^{j-i} \aleph_{2i-2}\wp_{2(n-j+1)} & \text{if } i \leq j \\ (-1)^{i+j} \frac{1}{2s\aleph_{2n}} t^{j-i} \aleph_{2j-2}\wp_{2(n-i+1)} & \text{if } i > j \end{cases}.$$

The sequence of the spectra of the matrices $E_{q,n}$ for $n = 2,3,4,5,6$, are given in the following

$S_2 = \{2.697224362268005,\ 6.302775637731995\}$

$S_3 = \{2.669941260432018,\ 5.201639675723405, 7.128419063844577\}$

$$S_4 = \left\{ \begin{array}{c} 2.667028328506892,\ 4.700596159221179, \\ 6.156443643217495,\ 7.475931869054436 \end{array} \right\}$$

$$S_5 = \left\{ \begin{array}{c} 2.666706814990114,\ 4.449448536406551, \\ 5.511997050564387,\ 6.720107682566076, \\ 7.651739915472869 \end{array} \right\},$$
$$S_6 = \left\{ \begin{array}{c} 2.666671127024368,\ 4.309895427899775, \\ 5.098757535396342,\ 6.105336258757300, \\ 7.067037793312944,\ 7.752301857609272 \end{array} \right\}.$$

Clearly, $\sum \lambda_i = tr(E_{q,n}) = n(4s^2+2t) - 2s^2 - t$ and $\prod \lambda_i = \det(E_{q,n}) = \aleph_{2n}$.

If we take care of the spectra, minimum eigenvalue converges to $2s^2$. The maximum eigenvalue of spectra converges to $4s^2+4t$.

1.9 Some properties of tridiagonal matrices $O_{q,n}$ by odd (s,t)-Modified Pell sequence

Assume that $O_{q,n}$ is an $n \times n$ tridiagonal matrix defined as

$$O_{q,n} = \begin{bmatrix} s & -s & & & \\ t & 4s^2+2t & t & & \\ & t & 4s^2+2t & \ddots & \\ & & \ddots & \ddots & t \\ & & & t & 4s^2+2t \end{bmatrix}.$$

Then the determinant of $O_{q,n}$

$$\det O_{q,n} = \aleph_{2n-1}.$$

For the inverse of $O_{q,n}$, the values are computed as

$$\begin{aligned} a_1 &= s \\ a_i &= 4s^2+2t,\ i \geq 2 \\ c_1 &= -s,\quad b_1 = t \\ b_i &= c_i = t,\quad i > 1 \\ \theta_0 &= 1,\quad \theta_i = \aleph_{2i-1},\quad i \geq 1 \\ \phi_j &= \frac{\wp_{2(n-j+2)}}{2s},\quad j \geq 1. \end{aligned}$$

If $s = t = 1$, then the sequence of the spectra of the matrices $O_{q,n}$ for $n = 2, 3, 4, 5, 6$ is given in the following

$S_2 = \{1.208712152522080,\ 5.791287847477920\}$

$S_3 = \{1.218716204021644,\ 4.862194798386097,\ 6.919088997592260\}$

$$S_4 = \left\{ \begin{array}{c} 1.219199198376559,\ 4.504733902179790 \\ 5.898455994015420,\ 7.377610905428227 \end{array} \right\}$$

$$S_5 = \left\{ \begin{array}{c} 1.219222421149124,\ 4.332385534359584, \\ 5.293995934620866,\ 6.555626057939926, \\ 7.598770051930496 \end{array} \right\}$$

$$S_6 = \left\{ \begin{array}{c} 1.219223537248886,\ 4.235965700260784, \\ 4.930440201174632,\ 5.932935651316058, \\ 6.960679940971054,\ 7.720754969028587 \end{array} \right\}.$$

The sequence of maximum eigenvalue is increasing and converges to $4s^2 + 4t = 8$. So we can say $\lim\limits_{n\to\infty} \max(\lambda(O_n(k)) = 4s^2 + 4t$.

Evidently, $\sum \lambda_i = tr(O_{q,n}) = (n-1)(4s^2 + 2t) + s$ and $\prod \lambda_i = \det(O_{q,n}) = \aleph_{2n-1}$.

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Coefficient Bounds for Al-Oboudi Type Bi-univalent Functions based on a Modified Sigmoid Activation Function and Horadam Polynomials

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Abstract

Using the Al-Oboudi type operator, we present and investigate two special families of bi-univalent functions in $\mathfrak{D}$, an open unit disc, based on $\phi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$, a modified sigmoid activation function (MSAF) and Horadam polynomials. We estimate the initial coefficients bounds for functions of the type $g_\phi(z) = z + \sum\limits_{j=2}^{\infty} \phi(s) d_j z^j$ in these families. Continuing the study on the initial cosfficients of these families, we obtain the functional of Fekete-Szegö for each of the two families. Furthermore, we present few interesting observations of the results investigated.

1 Preliminaries

Let the set of complex numbers be denoted by $\mathbb{C}$ and the set of normalized regular functions in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ that have the power series of the form

$$g(z) = z + d_2 z^2 + d_3 z^3 + ... = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1.1}$$

be indicated by $\mathcal{A}$ and the set of all functions of $\mathcal{A}$ that are univalent in $\mathfrak{D}$ is symbolized by $\mathcal{S}$. The famous Koebe theorem (see [12]) ensures that any function

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$g \in \mathcal{S}$ has an inverse g^{-1} satisfying $z = g^{-1}(g(z))$, $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$ and $r_0(g) \geq 1/4$, $z, \omega \in \mathfrak{D}$, where

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \ldots. \quad (1.2)$$

A function g of $\mathcal{A}$ is said to be bi-univalent (or bi-schlicht) in $\mathfrak{D}$ if g and its inverse g^{-1} are both univalent (or schlicht) in $\mathfrak{D}$. The set of bi-univalent functions having the form (1.1) is indicated by $\sum$. Historically investigations of the family $\sum$ begun five decades ago by Lewin [23] and Brannan et al. [9]. After few years, Tan [40] found the initial coefficient bounds of bi-univalent functions. Later, Brannan and Taha [10] presented and investigated certain subsets of $\sum$ similar to convex and starlike functions of order σ $(0 \leq \sigma < 1)$ in $\mathfrak{D}$. Some interesting results concerning initial bounds for certain special sets of $\sum$ have been appreared in [11], [18] and [32].

Let the set of real numbers be $\mathbb{R} = (-\infty, \infty)$ and the set positive integers be $\mathbb{N} := \mathbb{N}_0 \backslash \{0\} = \{1, 2, 3, \ldots\}$.

Recently, Hörzum and Koçer [21] (see also [20]) examined the Horadam polynomials $H_j(x)$ (or $H_j(x, a, b; p, q)$). It is given by the recurrence relation

$$H_j(x) = pxH_{j-1}(x) + qH_{j-2}(x), \ H_1(x) = a, \ \ H_2(x) = bx, \quad (1.3)$$

where $j \in \mathbb{N}\backslash\{1, 2\}$, $x \in \mathbb{R}$, p, q, a and b are real constants. It is easy to see from (1.3) that $H_3(x) = pbx^2 + qa$. The generating function of the sequence $H_j(x)$, $j \in \mathbb{N}$, is as below (see [21]):

$$\mathcal{G}(x, z) := \sum_{j=1}^{\infty} H_j(x)z^{j-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}, \quad (1.4)$$

where $z \in \mathbb{C}$ is independent of the argument $x \in \mathbb{R}$, that is $\Re(z) \neq x$.

Few particular cases of $H_j(x, a, b; p, q)$ are:

i) $H_j(x, 1, 1; 1, 1) = F_j(x)$, *ii)* $H_j(x, 1, 2; 2, -1) = U_j(x)$,
iii) $H_j(x, 1, 1; 2, -1) = T_j(x)$, *iv)* $H_j(x, 2, 1; 1, 1) = L_j(x)$,
v) $H_j(x, 2, 2; 2, 1) = Q_j(x)$ and *vi)* $H_j(x, 1, 2; 2, 1) = P_j(x)$.

They are named as Fibonacci polynomials, second type Chebyshev polynomials, first type Chebyshev polynomials, Lucas polynomials, Pell-Lucas polynomials and Pell polynomials, respectively.

In the literature, the estimates on $|d_2|$, $|d_3|$ and the famous inequality of Fekete-Szegö were determined for bi-univalent functions related to certain polynomials like Fibonacci polynomials, (p,q)-Lucas polynomials, second kind Chebyshev polynomials and Horadam polynomials. We also note that the above polynomials and other special polynomials are potentially important in statistical, physical, mathematical and engineering sciences. Additional information about these polynomials can be found in [7], [8], [16], [17], [24] and [42]. More details about the famous Fekete-Szegö problem connected with Haradam polynomials are available with the works of [1], [2], [3], [26], [31], [38] and [41].

The recent research trend is the study of bi-univalent functions linked with any one of the above mentioned polynomials using well-known operators, which can be seen in the research papers [4], [13], [25], [28], [34], [36], [37] and [39]. Generally interest was shown to estimate the initial Taylor-Maclaurin coefficients and the celebrated inequality of Fekete-Szegö for the special families of $\sum$ that are being introduced using known operators.

In this work, we present two special sets of $\sum$ using Al-Oboudi type operator which was precisely defined in the paper [19]. We determine the initial coefficient bounds and also obtain the relevant connection to the celebrated Fekete-Szegö functional for functions in the defined families.

Let $\mathcal{A}_\phi$ denote the set of regular functions of the form

$$g_\phi(z) = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j,$$

where $\phi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$, is a MSAF. Note that $\phi(0) = 1$ and hence $\mathcal{A}_1 := \mathcal{A}$ (see [14]).

Definition 1.1. For $g_\phi \in \mathcal{A}_\phi$, $k \in \mathbb{N}_0, \beta \geq 0$, an Al-Oboudi type operator D^k_β :

$\mathcal{A}_\phi \to \mathcal{A}_\phi$, is defined by

$$D_\beta^0 g_\phi(z) = g_\phi(z),\ D_\beta^1 g_\phi(z) = (1-\beta)g_\phi(z) + \beta z g'_\phi(z), ..., D_\beta^k g_\phi(z) = D_\beta(D_\beta^{k-1} g_\phi(z)),$$
$$z \in \mathfrak{D}.$$

Remark 1.1. If $g_\phi(z) = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j \in \mathcal{A}_\phi$, $z \in \mathfrak{D}$, then

$$D_\beta^k g_\phi(z) = z + \sum_{j=2}^{\infty} (1+(j-1)\beta)^k \phi(s) d_j z^j,\ z \in \mathfrak{D}.$$

When $\phi(s) = 1$, we get the Al-Oboudi operator [5], which reduces to the Sălăgean operator [29], if $\beta = 1$.

For regular functions g and f in $\mathfrak{D}$, g is said to subordinate to f, if there is a Schwarz function ψ in $\mathfrak{D}$, such that $\psi(0) = 0$, $|\psi(z)| < 1$ and $g(z) = f(\psi(z))$, $z \in \mathfrak{D}$. This subordination is indicated as $g \prec f$ or $g(z) \prec f(z)$. Specifically, when $f \in \mathcal{S}$ in $\mathfrak{D}$, then $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(\mathfrak{D}) \subset f(\mathfrak{D})$.

Inspired by the articles [6], [33] and the trends on functions $\in \sum$, we present two special families of $\sum$ by using Al-Oboudi type operator, which is as in Definition 1.1 and Horadam polynomials $H_j(x)$ as in the relation (1.3) having the generating function (1.4).

Throughout this paper, $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), p, q, a and b are as in (1.3) and $\mathcal{G}$ is as in (1.4), unless and otherwise mentioned.

Definition 1.2. A function g in $\sum$ having the power series (1.1) is said to be in the family $S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$, $0 \le \gamma \le 1$, $\mu \ge 0$, $k \in \mathbb{N}_0$, $\beta \ge 0$ and $\phi(s)$ the MSAF, if

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2 (D_\beta^k g_\phi(z))''}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} \prec 1 - a + \mathcal{G}(x, z),\ z \in \mathfrak{D}$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu \omega^2 (D_\beta^k f_\phi(\omega))''}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} \prec 1 - a + \mathcal{G}(x, \omega),\ \omega \in \mathfrak{D}.$$

We note that i) $\mu = 0$, ii) $\gamma = 0$ and iii) $\gamma = 1$ lead the family $\mathrm{S}\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$ to the below mentioned subfamilies:

1. $SK_{\sum}(x, \gamma, k, \beta, \phi(s)) \equiv S\mathfrak{S}_{\sum}(x, \gamma, 0, k, \beta, \phi(s))$ is the set of functions $g \in \sum$ satisfying

$$\frac{z(D_\beta^k g_\phi(z))'}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} \prec 1-a+\mathcal{G}(x,z), \text{ and } \frac{\omega(D_\beta^k f_\phi(\omega))'}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} \prec 1-a+\mathcal{G}(x,\omega),$$

where $z, \omega \in \mathfrak{D}$.

2. $SL_{\sum}(x, \mu, k, \beta, \phi(s)) \equiv S\mathfrak{S}_{\sum}(x, 0, \mu, k, \beta, \phi(s))$ is the family of functions $g \in \sum$ satisfying

$$(D_\beta^k g_\phi(z))' + \mu z(D_\beta^k g_\phi(z))'' \prec 1 - a + \mathcal{G}(x, z)$$

and

$$(D_\beta^k f_\phi(\omega))' + \mu\omega(D_\beta^k f_\phi(\omega))'' \prec 1 - a + \mathcal{G}(x, \omega),$$

where $z, \omega \in \mathfrak{D}$.

3. $SM_{\sum}(x, \mu, k, \beta, \phi(s)) \equiv S\mathfrak{S}_{\sum}(x, 1, \mu, k, \beta, \phi(s))$ is the family of functions $g \in \sum$ satisfying

$$\left(\frac{z(D_\beta^k g_\phi(z))'}{D_\beta^k g_\phi(z)}\right) + \mu\left(\frac{z(D_\beta^k g_\phi(z))''}{D_\beta^k g_\phi(z)}\right) \prec 1 - a + \mathcal{G}(x, z)$$

and

$$\left(\frac{\omega(D_\beta^k f_\phi(\omega))'}{D_\beta^k f_\phi(\omega)}\right) + \mu\left(\frac{\omega(D_\beta^k f_\phi(\omega))''}{D_\beta^k f_\phi(\omega)}\right) \prec 1 - a + \mathcal{G}(x, \omega),$$

where $z, \omega \in \mathfrak{D}$.

Letting $k = 0$ and $\phi(s) = 1$ in the Definition 1.2, we obtain the family $SN_{\sum}(x, \gamma, \mu) \equiv S\mathfrak{S}_{\sum}(x, \gamma, \mu, 0, \beta, 1)$ of functions $g \in \sum$ satisfying

$$\frac{zg'(z) + \mu z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} \prec 1-a+\mathcal{G}(x,z) \quad \text{and} \quad \frac{\omega f'(\omega) + \mu\omega^2 f''(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} \prec 1-a+\mathcal{G}(x,\omega),$$

where $z, \omega \in \mathfrak{D}$, $f(\omega) = g^{-1}(\omega)$ is as given by (1.2), a is as in (1.3) and $\mathcal{G}$ is as in (1.4).

Definition 1.3. A function $g \in \sum$ having the power series (1.1) is said to be in the family $S\mathfrak{B}_{\sum}(x, \gamma, \tau, k, \beta, \phi(s))$, $0 \leq \gamma \leq 1$, $\tau \geq 1$, $k \in \mathbb{N}_0$, $\beta \geq 0$ and $\phi(s)$ the MSAF, if

$$\frac{z[(D_\beta^k g_\phi(z))']^\tau}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} \prec 1 - a + \mathcal{G}(x, z),\ z \in \mathfrak{D}$$

and

$$\frac{\omega[(D_\beta^k f_\phi(\omega))']^\tau}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} \prec 1 - a + \mathcal{G}(x, \omega),\ \omega \in \mathfrak{D}.$$

Note that the certain choices of γ lead the family $S\mathfrak{B}_{\sum}(x, \gamma, \tau, k, \beta, \phi(s))$ to the following two subclasses:

1. $SP_{\sum}(x, \tau, k, \beta, \phi(s)) \equiv S\mathfrak{B}_{\sum}(x, 0, \tau, k, \beta, \phi(s))$ is the set of functions $g \in \sum$ satisfying

$$[(D_\beta^k g_\phi(z))']^\tau \prec 1-a+\mathcal{G}(x, z),\ z \in \mathfrak{D} \quad \text{and} \quad [(D_\beta^k f_\phi(\omega))']^\tau \prec 1-a+\mathcal{G}(x, \omega),\ \omega \in \mathfrak{D},$$

2. $S\mathfrak{N}_{\sum}(x, \tau, k, \beta, \phi(s)) \equiv S\mathfrak{B}_{\sum}(x, 1, \tau, k, \beta, \phi(s))$ is the class of functions $g \in \sum$ satisfying

$$\frac{z[(D_\beta^k g_\phi(z))']^\tau}{D_\beta^k g_\phi(z)} \prec 1-a+\mathcal{G}(x, z),\ z \in \mathfrak{D} \text{ and } \frac{\omega[(D_\beta^k f_\phi(\omega))']^\tau}{D_\beta^k f_\phi(\omega)} \prec 1-a+\mathcal{G}(x, \omega),\ \omega \in \mathfrak{D},$$

$S\mathfrak{N}_{\sum}(x, \tau, k, \beta, \phi(s))$ is the family of Al-Oboudi type τ-bi-pseudo-starlike functions associated with Horadam polynomials involving the MSAF.

On taking $k = 0$ and $\phi(s) = 1$ in Definition 1.3, we get the family $SQ_{\sum}(x, \gamma, \tau) \equiv S\mathfrak{B}_{\sum}(x, \gamma, \tau, 0, \beta, 1)$ of functions $g(z)$ in $\sum$ satisfying

$$\frac{z(g'(z))^\tau}{\gamma g(z) + (1-\gamma)z} \prec 1 - a + \mathcal{G}(x, z) \quad \text{and} \quad \frac{\omega(f'(\omega))^\tau}{\gamma f(\omega) + (1-\gamma)\omega} \prec 1 - a + \mathcal{G}(x, \omega),$$

where $z, \omega \in \mathfrak{D}$, $f(\omega) = g^{-1}(\omega)$ is as given by (1.2), a is as in (1.3) and $\mathcal{G}$ is as in (1.4).

Remark 1.2. We note that $i)$ $S\mathfrak{B}_{\sum}(x, \gamma, 1, k, \beta, \phi(s)) \equiv SK_{\sum}(x, \gamma, k, \beta, \phi(s))$, ii) $S\mathfrak{N}_{\sum}(x, 1, k, \beta, \phi(s)) \equiv SK_{\sum}(x, 1, k, \beta, \phi(s)) \equiv SM_{\sum}(x, 0, k, \beta, \phi(s))$ and iii) $SP_{\sum}(x, 1, k, \beta, \phi(s)) \equiv SK_{\sum}(x, 0, k, \beta, \phi(s)) \equiv SL_{\sum}(x, 0, k, \beta, \phi(s))$.

Remark 1.3. i) For $\mu = \gamma = 0$, the class $SN_{\Sigma}(x, 0, 0) \equiv \mathcal{H}_{\Sigma}(x)$ was studied by Alamoush [2] and ii) For $\mu = 0$ and $\gamma = 1$, the family $SN_{\Sigma}(x, 1, 0) \equiv S^*_{\Sigma}(x)$ was investigated by Srivastava et al. [32].

Remark 1.4. i) For $\gamma = 0$, the family $SQ_{\Sigma}(x, 0, \tau) \equiv S^*_{\Sigma}(x, \tau)$ was investigated by Abirami et al. [1] and ii) For $\beta = 1$, the family $S\mathfrak{N}_{\sum}(x, \tau, k, 1, \phi(s)) \equiv M_{\Sigma}(x, \tau, k, \phi(s))$ was considered in [25].

Remark 1.5. In a special situation, if we choose $a = 1, b = p = 2, q = -1$ and $x \to t$, the generating function (1.4) reduces to the second type Chebyshev polynomials $U_j(t)$, which is explicitly given by

$$U_j(t) = (j+1)\ {}_2F_1\left(-j, j+2; \frac{3}{2}; \frac{1-t}{2}\right) = \frac{sin(j+1)\psi}{sin\psi},\ (t = sin\psi)$$

in terms of the Gauss hypergeometric function ${}_2F_1$. In this particular situation, the bi-univalent function families $S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$ and $S\mathfrak{B}_{\sum}(x, \gamma, \tau, k, \beta, \phi(s))$ would become the families $S\mathfrak{S}_{\sum}(t, \gamma, \mu, k, \beta, \phi(s))$ and $S\mathfrak{B}_{\sum}(t, \gamma, \tau, k, \beta, \phi(s))$, respectively. The families $S\mathfrak{B}_{\sum}(t, 1, \tau, 0, \beta, \phi(s)) \equiv AO_{\sum}(t, \tau, \phi(s))$ and $S\mathfrak{B}_{\sum}(t, 1, \tau, 0, \beta, 1) \equiv AY_{\sum}(t, \tau)$ were studied earlier in [8] and [7], respectively.

In Section 2, we derive the estimates for $|d_2|$, $|d_3|$ and the inequality of Fekete-Szegö [15] for functions of the form (1.1) $\in S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$ and we also present some observations of our result. In Section 3, we derive the estimates for $|d_2|$, $|d_3|$ and the Fekete-Szegö inequality for functions of the form (1.1) $\in S\mathfrak{B}_{\sum}(x, \gamma, \tau, k, \beta, \phi(s))$. Few interesting consequences of the result are also considered.

2 Estimates for Function Family $S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$

In the following theorem, we determine the initial coefficients bounds and the inequality of Szegö for functions in $S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$.

Theorem 2.1. *Let* $0 \leq \gamma \leq 1, \mu \geq 0, k \in \mathbb{N}_0, \beta \geq 0$ *and* $\phi(s)$ *be the MSAF. If the function* $g \in S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$, *then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k \phi(s)\sqrt{|(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(bx)^2-(2(\mu+1)-\gamma)^2(pbx^2+qa)|}}, \tag{2.1}$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k \phi(s)}\left[\frac{(bx)^2}{(2(\mu+1)-\gamma)^2} + \frac{|bx|}{(3(2\mu+1)-\gamma)}\right] \tag{2.2}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{(1+2\beta)^k \phi(s)(3(2\mu+1)-\gamma)} & ; \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| \leq J \\ \frac{|bx|^3 \left|1-\frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k \phi(s)|(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(bx)^2-(2(\mu+1)-\gamma)^2(pbx^2+qa)|} & ; \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| \geq J, \end{cases} \tag{2.3}$$

where

$$J = \frac{1}{(3(2\mu+1)-\gamma)}\left|\gamma^2 - (2\mu+3)\gamma + 3(2\mu+1) - (2(\mu+1)-\gamma)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|. \tag{2.4}$$

Proof. Let $g \in S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$. Then, for two regular functions $\mathfrak{M}$, $\mathfrak{N}$ with $\mathfrak{M}(0) = 0, |\mathfrak{M}(z)| < 1, \mathfrak{N}(0) = 0$ and $|\mathfrak{N}(\omega)| < 1$, $z, \omega \in \mathfrak{D}$ and on account of Definition 1.2, we can write

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2 (D_\beta^k g_\phi(z))''}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} = 1 - a + \mathcal{G}(x, \mathfrak{M}(z))$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu \omega^2 (D_\beta^k f_\phi(\omega))''}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} = 1 - a + \mathcal{G}(x, \mathfrak{N}(\omega)).$$

Or, equivalently

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2 (D_\beta^k g_\phi(z))''}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} = 1 - a + H_1(x) + H_2(x)\mathfrak{m}(z) + H_3(x)(\mathfrak{m}(z))^2 + ... \tag{2.5}$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu\omega^2(D_\beta^k f_\phi(\omega))''}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} = 1 - a + H_1(x) + H_2(x)\mathfrak{n}(\omega) + H_3(x)(\mathfrak{n}(\omega))^2 + \ldots. \tag{2.6}$$

From (2.5) and (2.6), in view of (1.3), we find

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2(D_\beta^k g_\phi(z))''}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} = 1 + H_2(x)\mathfrak{m}_1 z + [H_2(x)\mathfrak{m}_2 + H_3(x)\mathfrak{m}_1^2]z^2 + \ldots \tag{2.7}$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu\omega^2(D_\beta^k f_\phi(\omega))''}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} = 1 + H_2(x)\mathfrak{n}_1\omega + [H_2(x)\mathfrak{n}_2 + H_3(x)\mathfrak{n}_1^2]\omega^2 + \ldots. \tag{2.8}$$

It is well known that if $|\mathfrak{M}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + \ldots| < 1$, $z \in \mathfrak{D}$ and $|\mathfrak{N}(\omega)| = |\mathfrak{n}_1\omega + \mathfrak{n}_2\omega^2 + \mathfrak{n}_3\omega^3 + \ldots| < 1$, $\omega \in \mathfrak{D}$, then

$$|\mathfrak{m}_i| \leq 1 \,\text{and}\, |\mathfrak{n}_i| \leq 1 \,(i \in \mathbb{N}). \tag{2.9}$$

We easily get the following by equating the corresponding coefficients in (2.7) and (2.8):

$$(1+\beta)^k\phi(s)\,(2(\mu+1)-\gamma)d_2 = H_2(x)\mathfrak{m}_1 \tag{2.10}$$

$$(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)d_3 - (1+\beta)^{2k}\phi^2(s)(2(\mu+1)-\gamma)\gamma\, d_2^2 = H_2(x)\mathfrak{m}_2 + H_3(x)\mathfrak{m}_1^2 \tag{2.11}$$

$$-(1+\beta)^k\phi(s)\,(2(\mu+1)-\gamma)\,d_2 = H_2(x)\mathfrak{n}_1 \tag{2.12}$$

$$\begin{aligned}-(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)d_3 + (1+\beta)^{2k}\phi^2(s)(\gamma^2 - 2(\mu+2)\gamma + 6(2\mu+1))d_2^2 \\ = H_2(x)\mathfrak{n}_2 + H_3(x)\mathfrak{n}_1^2.\end{aligned} \tag{2.13}$$

From (2.10) and (2.12), we easily obtain

$$\mathfrak{m}_1 = -\mathfrak{n}_1 \tag{2.14}$$

and also

$$2(1+\beta)^{2k}\phi^2(s)(2(\mu+1)-\gamma)^2 d_2^2 = (\mathfrak{m}_1^2 + \mathfrak{n}_1^2)(H_2(x))^2. \tag{2.15}$$

If we add (2.11) and (2.13), then we obtain

$$2(1+\beta)^{2k}\phi^2(s)(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))d_2^2 = H_2(x)(\mathfrak{m}_2+\mathfrak{n}_2)+H_3(x)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \tag{2.16}$$

Substituting the value of $\mathfrak{m}_1^2+\mathfrak{n}_1^2$ from (2.15) in (2.16), we get

$$d_2^2 = \frac{(H_2(x))^3(\mathfrak{m}_2+\mathfrak{n}_2)}{2(1+\beta)^{2k}\phi^2(s)\left[(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(h_2(x))^2-(2(\mu+1)-\gamma)^2h_3(x)\right]}, \tag{2.17}$$

which yields (2.1) on using (2.9).

After subtracting (2.13) from (2.11) and then using (2.14), we obtain

$$d_3 = \frac{(1+\beta)^{2k}\phi(s)}{(1+2\beta)^k}d_2^2 + \frac{H_2(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)}. \tag{2.18}$$

Then in view of (2.15), (2.18) becomes

$$d_3 = \frac{(H_2(x))^2(\mathfrak{m}_1^2+\mathfrak{n}_1^2)}{2(1+2\beta)^k\phi(s)(2(\mu+1)-\gamma)^2} + \frac{H_2(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)},$$

which yields (2.2) on using (2.9).

From (2.17) and (2.18), for $\delta \in \mathbb{R}$, we get

$$|d_3-\delta d_2^2| = |H_2(x)|\left|\left(T(\delta,x)+\frac{1}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)}\right)\mathfrak{m}_2+\left(T(\delta,x)-\frac{1}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)}\right)\mathfrak{n}_2\right|,$$

where

$$T(\delta,x) = \frac{\left(\frac{(1+\beta)^{2k}\phi(s)}{(1+2\beta)^k}-\delta\right)(H_2(x))^2}{2(1+\beta)^{2k}\phi^2(s)\left[(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(H_2(x))^2-(2(\mu+1)-\gamma)^2H_3(x)\right]}.$$

In view of (1.3), we conclude that

$$|d_3-\delta d_2^2| \le \begin{cases} \frac{|H_2(x)|}{(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)} & ;\ 0\le |T(\delta,x)| \le \frac{1}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)} \\ 2|H_2(x)||T(\delta,x)| & ;\ |T(\delta,x)| \ge \frac{1}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)}, \end{cases}$$

which gets (2.3) with J as in (2.4). This evidently ends the proof of Theorem 2.1. □

By setting i) $\mu = 0$, ii) $\gamma = 0$, iii) $\gamma = 1$ and iv) $k = 0, \phi(s) = 1$ in Theorem 2.1, we have the following four corollaries, respectively.

Corollary 2.1. *If the function $g \in SK_{\sum}(x, \gamma, k, \beta, \phi(s))$, then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k\phi(s)\sqrt{|(\gamma^2-3\gamma+3)(bx)^2-(2-\gamma)^2(pbx^2+qa)|}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{b^2x^2}{(2-\gamma)^2}+\frac{|bx|}{3-\gamma}\right]$$

and for some $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{(1+2\beta)^k\phi(s)(3-\gamma)} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq J_1 \\ \frac{|bx|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)|(\gamma^2-3\gamma+3)(bx)^2-(2-\gamma)^2(pbx^2+qa)|} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq J_1, \end{cases}$$

where $J_1 = \frac{1}{(3-\gamma)}\left|\gamma^2-3\gamma+3-(2-\gamma)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|.$

Remark 2.1. For $\gamma = \beta = 1$, Corollary 2.1 reduce to Corollary 2.1 of Magesh et al. [25] and Corollary 2.1 further coincide with Corollary 2.1 of Abirami et al. [1], when $k = 0$ and $\phi(s) = 1$. Corollary 2.1 coincide with Theorem 2.2 of Alamoush [3], when $\gamma = k = 0$ and $\phi(s) = 1$ and also we obtain Corollary 1 and Corollary 3 of [31] for $k = 0$, $\gamma = \phi(s) = 1$.

Corollary 2.2. *If the function $g \in SL_{\sum}(x, \gamma, k, \beta, \phi(s))$, then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k\phi(s)\sqrt{|3(2\mu+1)(bx)^2-4(\mu+1)^2(pbx^2+qa)|}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{b^2x^2}{4(\mu+1)^2}+\frac{|bx|}{3(2\mu+1)}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu+1)(1+2\beta)^k\phi(s)} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq J_2 \\ \frac{|bx|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)|3(2\mu+1)(bx)^2-4(\mu+1)^2(pbx^2+qa)|} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq J_2, \end{cases}$$

where $J_2 = \left|1-\frac{4(\mu+1)^2}{3(2\mu+1)}\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|.$

Remark 2.2. For $\mu = k = 0$ and $\phi(s) = 1$ Corollary 2.2 coincide with Theorem 2.2 of [3].

Corollary 2.3. *If the function $g \in SM_{\sum}(x, \mu, k, \beta, \phi(s))$, then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k \phi(s)\sqrt{\left[|(4\mu+1)(bx)^2 - (2\mu+1)^2(pbx^2+qa)|\right]}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{b^2x^2}{(2\mu+1)^2} + \frac{|bx|}{2(3\mu+1)}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{2(3\mu+1)(1+2\beta)^k\phi(s)} & ; \left|1 - \frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq J_3 \\ \frac{|bx|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)[(4\mu+1)(bx)^2-(2\mu+1)^2(pbx^2+qa)]} & ; \left|1 - \frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq J_3, \end{cases}$$

where $J_3 = \frac{1}{2(3\mu+1)}\left|(4\mu+1) - (2\mu+1)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Remark 2.3. Corollary 2.3 coincide with Theorem 2.1 of Magesh et al. [26], when $k = 0$ and $\phi(s) = 1$. Also we obtain Corollary 2.1 of [25] from Corollary 2.3, when $\mu = 0$ and $\beta = 1$.

Corollary 2.4. *If the function $g(z) \in SN_{\sum}(x, \gamma, \mu)$, then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(\gamma^2 - (2\mu+3)\gamma + 3(2\mu+1))(bx)^2 - (2(\mu+1)-\gamma)^2(pbx^2+qa)|}},$$

$$|d_3| \leq \frac{(bx)^2}{(2(\mu+1)-\gamma)^2} + \frac{|bx|}{(3(2\mu+1)-\gamma)}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu+1)-\gamma} & ; |1-\delta| \leq J_4 \\ \frac{|bx|^3\,|1-\delta|}{|(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(bx)^2-(2(\mu+1)-\gamma)^2(pbx^2+qa)|} & ; |1-\delta| \geq J_4, \end{cases}$$

where

$$J_4 = \frac{1}{(3(2\mu+1)-\gamma)}\left|\gamma^2 - (2\mu+3)\gamma + 3(2\mu+1) - (2(\mu+1)-\gamma)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|.$$

Remark 2.4. By choosing appropriate values for parameters γ and μ in Corollary 2.4, we obtain Theorem 2.2, Theorem 2.1 and Corollaries 1, 2 of [3], [26] and [31], respectively, as it can be seen from earlier remarks.

3 Estimates for the Function Family $S\mathfrak{B}_{\sum}(x, \gamma, \tau, k,\ \beta, \phi(s))$

In the next theorem, we find the first two Taylor-Maclaurin coefficients and the inequality of Fekete-Szegö for functions in $S\mathfrak{B}_{\sum}(x, \gamma, \tau, k,\ \beta, \phi(s))$.

Theorem 3.1. *Let* $0 \leq \gamma \leq 1$, $\tau \geq 1$, $k \in \mathbb{N}_0$, $\beta \geq 0$ *and* $\phi(s)$ *the MSAF. If the function* $g \in S\mathfrak{B}_{\sum}(x,\ \gamma,\ \tau, k,\ \beta, \phi(s))$, *then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k \phi(s)\sqrt{|(\gamma^2 + (\tau - \gamma)(2\tau + 1))(bx)^2 - (2\tau - \gamma)^2(pbx^2 + qa)|}}, \tag{3.1}$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k \phi(s)} \left[\frac{(bx)^2}{(2\tau - \gamma)^2} + \frac{|bx|}{(3\tau - \gamma)}\right] \tag{3.2}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{(1+2\beta)^k \phi(s)(3\tau - \gamma)} & ;\ |1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}| \leq \Omega \\ \frac{\left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| |bx|^3}{(1+2\beta)^k \phi(s) |(\gamma^2 + (\tau-\gamma)(2\tau+1))(bx)^2 - (2\tau-\gamma)^2(pbx^2+qa)|} & ;\ |1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}| \geq \Omega, \end{cases} \tag{3.3}$$

where

$$\Omega = \frac{1}{(3\tau - \gamma)} \left|(\gamma^2 + (\tau - \gamma)(2\tau + 1)) - (2\tau - \gamma)^2 \left(\frac{pbx^2 + qa}{b^2 x^2}\right)\right|.$$

Proof. Let $g \in S\mathfrak{B}_{\sum}(x, \gamma,\ \tau, k,\ \beta, \phi(s))$. Then, for some regular functions $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathfrak{M}(0) = 0, |\mathfrak{M}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + ...| < 1, \mathfrak{N}(0) = 0$ and $|\mathfrak{N}(\omega)| = |\mathfrak{n}_1 \omega + \mathfrak{n}_2 \omega^2 + \mathfrak{n}_3 \omega^3 + ...| < 1$, $z, \omega \in \mathfrak{D}$ and on account of Definition 1.3, we can write

$$\frac{z[(D_\beta^k g_\phi(z))']^\tau}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} = 1 - a + \mathcal{G}(x, \mathfrak{M}(z)),\ z \in \mathfrak{D}$$

and

$$\frac{\omega[(D_\beta^k f_\phi(\omega))']^\tau}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} = 1 - a + \mathcal{G}(x, \mathfrak{N}(\omega)),\ \omega\ \in \mathfrak{D}.$$

Following the procedure similar to the proof of Theorem 2.1, one gets

$$(1+\beta)^k (2\tau - \gamma)\phi(s) d_2 = H_2(x)\mathfrak{m}_1 \tag{3.4}$$

$$(1+\beta)^{2k}\phi^2(s)(\gamma^2-2\tau\gamma+2\tau(\tau-1))d_2^2+(1+2\beta)^k\phi(s)(3\tau-\gamma)d_3 = H_2(x)\mathfrak{m}_2+H_3(x)\mathfrak{m}_1^2 \tag{3.5}$$

$$-(1+\beta)^k(2\tau-\gamma)\phi(s)d_2 = H_2(x)\mathfrak{n}_1 \tag{3.6}$$

$$\begin{aligned}&(1+\beta)^{2k}\phi^2(s)(\gamma^2-2(\tau+1)\gamma+2\tau(\tau+2))d_2^2-(1+2\beta)^k\phi(s)(3\tau-\gamma)d_3\\&= H_2(x)\mathfrak{n}_2+H_3(x)\mathfrak{n}_1^2.\end{aligned} \tag{3.7}$$

The results (3.1)-(3.3) now follow from (3.4)-(3.7) by adopting the procedure as in Theorem 2.1. □

By setting i) $\gamma=0$, ii) $\gamma=1$ and iii) $k=0$, $\phi(s)=1$ in Theorem 3.1, we have the following three corollaries.

Corollary 3.1. *If the function* $g \in SP_{\sum}(x,\tau,k,\ \beta,\phi(s))$, *then*

$$|d_2| \le \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k\phi(s)\sqrt{|\tau(2\tau+1)(bx)^2-4\tau^2(pbx^2+qa)|}},$$

$$|d_3| \le \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{(bx)^2}{4\tau^2}+\frac{|bx|}{3\tau}\right]$$

and for $\delta\in\mathbb{R}$,

$$|d_3-\delta d_2^2| \le \begin{cases}\frac{|b(x)|}{3\tau(1+2\beta)^k\phi(s)} & ;\ \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \le \Omega_1\\ \frac{\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right||bx|^3}{(1+2\beta)^k\phi(s)|\tau(2\tau+1)(bx)^2-4\tau^2(pbx^2+qa)|} & ;\ \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \ge \Omega_1,\end{cases}$$

where $\Omega_1=\frac{1}{3}\left|(2\tau+1)-4\tau\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Remark 3.1. Corollary 3.1 coincides with Theorem 2.1 of [3], when $k=0$ and $\tau=\phi(s)=1$.

Corollary 3.2. *If the function* $g \in S\mathfrak{N}_{\sum}(x,\tau,k,\ \beta,\phi(s))$, *then*

$$|d_2| \le \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k\phi(s)\sqrt{|(\tau(2\tau-1))(bx)^2-(2\tau-1)^2(pbx^2+qa)|}},$$

$$|d_3| \le \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{(bx)^2}{(2\tau-1)^2}+\frac{|bx|}{(3\tau-1)}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{(1+2\beta)^k \phi(s)(3\tau-1)} & ; \ \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| \leq \Omega_2 \\ \frac{\left|1-\frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| |bx|^3}{(1+2\beta)^k \phi(s)|(\tau(2\tau-1))(bx)^2-(2\tau-1)^2(pbx^2+qa)|} & ; \ \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| \geq \Omega_2, \end{cases}$$

where $\Omega_2 = \frac{1}{(3\tau-1)} \left|(\tau(2\tau-1)) - (2\tau-1)^2 \left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Remark 3.2. Corollary 3.2 reduces to Theorem 2.1 of [25], when $\beta = 1$ and also the results of Corollary 3.2 coincide with Theorem 2.1 of Abirami et al. [1], when $k = 0$ and $\phi(s) = 1$.

Corollary 3.3. *If the function* $g \in SQ_{\sum}(x, \gamma, \tau)$, *then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(\gamma^2 + (\tau-\gamma)(2\tau+1))(bx)^2 - (2\tau-\gamma)^2(pbx^2+qa)|}},$$

$$|d_3| \leq \frac{(bx)^2}{(2\tau-\gamma)^2} + \frac{|bx|}{(3\tau-\gamma)}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{3\tau-\gamma} & ; |1-\delta| \leq \Omega_3 \\ \frac{|1-\delta||bx|^3}{|(\gamma^2+(\tau-\gamma)(2\tau+1))(bx)^2-(2\tau-\gamma)^2(pbx^2+qa)|} & ; |1-\delta| \geq \Omega_3, \end{cases}$$

where

$$\Omega_3 = \frac{1}{(3\tau-\gamma)} \left|(\gamma^2 + (\tau-\gamma)(2\tau+1)) - (2\tau-\gamma)^2 \left(\frac{pbx^2+qa}{b^2x^2}\right)\right|.$$

Remark 3.3. Corollary 3.3 reduces to Theorem 2.1 of [1], when $\gamma = 1$.

4 Conclusion

Two special families of holomorphic and bi-univalent (or bi-schlicht) functions are introduced by using Al-Oboudi type operator involving a modified sigmoid activation function associated with Horadam polynomials. Bounds of the first

two coefficients $|d_2|$, $|d_3|$ and the celebrated Fekete-Szegö functional have been fixed for each of the two families. Through corollaries of our main results, we have highlighted many interesting new consequences.

The special families examined in this research paper using Al-Oboudi type operator could inspire further research related to other aspects such as families using q-derivative operator [22], [35], meromorphic bi-univalent function families associated with Al-Oboudi differential operator [30] and families using integro-differential operators [27].

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The New Results in n-injective Modules and n-projective Modules

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Abstract

In this paper, we introduce and clarify a new presentation between the n-exact sequence and the n-injective module and n-projective module. Also, we obtain some new results about them.

1 Introduction

Category theory formalizes mathematical structures and their concepts in terms of a labeled directed graph called a category, whose nodes are called objects, and their edges called arrows (or morphisms). This category has two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. The language of category theory has been employed to formalize concepts of other high-level abstractions such as sets, rings, and groups. Several terms were utilized in category theory, including the âmorphismâ that is used differently from their usage in the rest of mathematics. In category theory, morphisms obey specific conditions of theory. Samuel Eilenberg and Saunders Mac Lane introduced the concepts of categories, functors, and natural transformations

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in 1942-45 in their study of algebraic topology, to understand the processes that preserve the mathematical structure. Category theory has practical applications in programming language theory, for example, the usage of monads in functional programming. It may also be used as an axiomatic foundation for mathematics, as an alternative to set theory and other proposed foundations. In mathematics, an abelian category is a category in which morphisms and objects can be added and in which kernels and cokernels exist and have desirable properties. The motivating prototype example of an abelian category is the category of abelian groups, Ab. The theory originated to unify several cohomology theories by Alexander Grothendieck and independently in the slightly earlier work of David Buchsbaum. Abelian categories are very stable categories. For example, they are regular and satisfy the snake lemma. The class of Abelian categories is closed under several categorical constructions, for instance, the category of chain complexes of an Abelian category, or the category of functors from a small category to an Abelian category are Abelian as well. These stability properties make them inevitable in homological algebra and beyond. This theory has significant applications in algebraic geometry, cohomology, and pure category theory. The Abelian categories are named after Niels Henrik Abel. An exact sequence is a concept in mathematics, especially in group theory, ring, and module theory, homological algebra, as well as in differential geometry. An exact sequence is a sequence, either finite or infinite, of objects and morphisms between them such that the image of one morphism equals the kernel of the next. Homological algebra is the branch of mathematics that studies homology in a general algebraic setting. It is a relatively young discipline, whose origins can be traced to investigations in combinatorial topology (a precursor to algebraic topology) and abstract algebra (theory of modules and syzygies) at the end of the 19th century, chiefly by Henri Poincaré and David Hilbert. The development of homological algebra has closely intertwined with the emergence of category theory. By and large, homological algebra is the study of homological functors and the intricate algebraic structures that they entail. One quite useful and ubiquitous concept in mathematics is that of chain complexes, which can be studied both through their homology and cohomology. Homological algebra affords the means to extract information

contained in these complexes and present it in the form of homological invariants of rings, modules, topological spaces, and other âtangibleâ mathematical objects. A powerful tool for doing this is provided by spectral sequences. From its very origins, homological algebra has played an enormous role in algebraic topology. Its sphere of influence has gradually expanded and presently includes commutative algebra, algebraic geometry, algebraic number theory, representation theory, mathematical physics, operator algebras, complex analysis, and the theory of partial differential equations. K-theory is an independent discipline that draws upon methods of homological algebra, as does the noncommutative geometry of Alain Connes. This paper is organized as follows.

In this paper, we show to prove the important theorems of n-injective modules and n-projective modules. Finally, we recall the definition of n-projective module, and we give an open problem about some theorems of n-projective modules.

2 Preliminaries

All rings R in this paper are assumed to have an identity element 1 (or unit) (where $r1 = r = 1r$ for all $r \in R$). We do not insist that $1 \neq 0$; however, should $1 = 0$, then R is the zero ring having only one element.

In this section, we recall some of the fundamental concepts and definitions, which are necessary for this paper. For details, we refer to [4,6,7,9,10,11].

Definition 2.1. An R-module M is injective provided that for every R-monomorphism $g : A \longrightarrow B$ between R-modules, any R-homomorphism $f : A \longrightarrow M$ can be extended to an R-homomorphism $h : B \longrightarrow M$ such that $hg = f$; i.e., the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"] & B \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

Definition 2.2. An R-module P is projective provided that for every R-epimorphism $g : A \longrightarrow B$ between R-modules and R-homomorphism f :

$P \longrightarrow B$, there exists an R-homomorphism $f : P \longrightarrow B$, there exists an R-homomorphism $h : P \longrightarrow A$ such that $gh = f$; i.e., the following diagram commutes

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "f"] & \\ A \arrow[r, "g"'] & B \arrow[r] & 0 \end{tikzcd}$$

Definition 2.3. A left R-module F is a free left R-module if F is isomorphic to a direct sum of copies of R; that is, there is a (possibly infinite) index set B with $F = \oplus_{b \in B} R_b$, where $R_b = \langle b \rangle \cong R$ for all $b \in B$. We call B a basis of F.

Definition 2.4. Let M be an R-module. An element $m \in M$ is divisible provided that for any $r \in R$ that is not a righ zero-divisor, there exists an $x \in M$ such that $m = rx$. We also say that M is a divisible module module provided that every element of M is divisible. Note that a divisible group is a divisible $\mathbb{Z}$-module.

Definition 2.5. Let $\mathcal{C}$ be an additive category and $f : A \longrightarrow B$ a morphism in $\mathcal{C}$. A weak cokernel of f is a morphism $g : B \longrightarrow C$ such that for all $C' \in \mathcal{C}$ the sequence of abelian groups

$$\mathcal{C}(C, C') \xrightarrow{g^*} \mathcal{C}(B, C') \xrightarrow{f^*} \mathcal{C}(A, C')$$

Definition 2.6. A category $\mathcal{C}$ is abelian if

1. $\mathcal{C}$ has a zero object.

2. For every pair of objects there is a product and a sum.

3. $\mathcal{C}$ Every map has a kernel and cokernel.

4. $\mathcal{C}$ Every monomorphism is a kernel of a map.

5. $\mathcal{C}$ Every epimorphism is a cokernel of a map.

Definition 2.7. A category $\mathcal{C}$ is additive if

1. $Hom(A,B)$ is an (additive) abelian group for every $A,B \in obj(\mathcal{C})$

2. the distributive laws hold: given morphisms

$$X \xrightarrow{a} A \xrightarrow{f} B \xrightarrow{b} Y$$

and

$$X \xrightarrow{a} A \xrightarrow{g} B \xrightarrow{b} Y$$

where X and $Y \in obj(\mathcal{C})$, then

$$b(f+g) = bf + bg$$

and $a \leftrightarrow [under]overb$

$$(f+g)a = fa + ga$$

3. $\mathcal{C}$ has a zero object.

4. $\mathcal{C}$ has finite product and finite coproduct.

Definition 2.8. An abelian group D is said to be divisible if given any $y \in D$ and $0 \neq n \in \mathbb{Z}$, there exists $x \in D$ such that $nx = y$.

Example 2.9.

1. *Note that Q is a divisible Z-module since for every $q \in Q$, where $q = \frac{a}{b}$ for integers $a, b \in Z$ with $b \neq 0$, and for every $0 \neq z \in Z$, there exists $x \in Q$ such that $x = \frac{a}{zb}$ so that $q = zx$.*

2. *Note that Z is not a divisible Z-module since there is no $x \in Z$ with $3 = 2x$.*

Definition 2.10. Let $M_{i i\in\mathbb{Z}}$ be a family of R-modules, and let $f_{i i\in\mathbb{Z}}$ be a family of R-homomorphisms such that $M_{i-1} \xrightarrow{f_i} Mi$ for every $i \in \mathbb{Z}$. Then the sequence

$$... \xrightarrow{f_{-1}} M_{-1} \xrightarrow{f_0} M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} ... \tag{2.1}$$

is said to be exact provided that $Im(fi-1) = Ker(f_i)$ for every $i \in \mathbb{Z}$. Note that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.2}$$

is exact if and only if f is an R-monomorphism, g is an R-epimorphism, and

$$Im(f) = Ker(g)$$

This type of sequence is called short exact.

Definition 2.11. Let $\mathcal{C}$ be an additive category and $f : A \longrightarrow B$ a morphism in $\mathcal{C}$. A weak cokernel of f is a morphism $g : B \longrightarrow C$ such that for all $C' \in \mathcal{C}$ the sequence of abelian groups

$$\mathcal{C}(C, C') \xrightarrow{\hat{g}} \mathcal{C}(B, C') \xrightarrow{\hat{f}} \mathcal{C}(A, C')$$

is exact. Equivalently, g is a weak cokernel of f if $fg = 0$ and for each morphism $h : B \longrightarrow C'$ such that $fh = 0$ there exists a (not necessarily unique) morphism $p : C \longrightarrow C'$ such that $h = gp$. These properties are subsumed in the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "g"] \arrow[d, "\forall h"] & C \arrow[dl, dashed, "\exists p"] \\
& C' &
\end{tikzcd}$$

Clearly, a weak cokernel g of f is a cokernel of f if and only if g is an epimorphism. The concept of weak kernel is defined dually.

Definition 2.12. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ is called $\mathcal{X}$-monic if

$$\mathcal{C}(B, X) \xrightarrow{\mathcal{C}(f,X)} \mathcal{C}(A, X) \longrightarrow 0$$

is exact for any object $X \in \mathcal{X}$. A morphism $f : A \longrightarrow X$ in $\mathcal{C}$ is called a left $\mathcal{X}$-approximation of A if f is $\mathcal{X}$-monic and $X \in \mathcal{X}$. The subcategory $\mathcal{X}$ is said to be a covariantly finite subcategory of $\mathcal{C}$ if any object A of $\mathcal{C}$ has a left $\mathcal{X}$-approximation. We can defined $\mathcal{X}$-epic morphism, right $\mathcal{X}$- approximation and contravariantly finite subcategory dually. The subcategory $\mathcal{X}$ is called functorially finite if it is both contravariantly finite and covariantly finite.

Definition 2.13. Let $\mathcal{C}$ be an additive category and $d^0 : X^0 \longrightarrow X^1$ a morphism in $\mathcal{C}$. An n-coker of d^0 is a sequence

$$(d^1, ..., d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^n} X^{n+1}$$

such that, for all $Y \in \mathcal{C}$ the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{C}(X^{n+1}, Y) \xrightarrow{\hat{d^n}} \mathcal{C}(X^n, Y) \xrightarrow{\hat{d^{n-1}}} ... \xrightarrow{\hat{d^1}} \mathcal{C}(X^1, Y) \xrightarrow{\hat{d^0}} \mathcal{C}(X^0, Y)$$

is exact. Equivalently, the sequence $(d^1, ..., d^n)$ is an n-coker of d^0 if, for all $1 \leq k \leq n-1$ the morphism d^k is a weak cokernel of d^{k-1}, and d^n is moreover a cokernel of d^{n-1}. In this case, we say the sequence

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^n} X^{n+1}$$

is right n-exact.

Remark 2.14. When we say n-cokernel we always means that n is a positive integer. We note that the notion of 1-cokernel is the same as cokernel. we can define n- kernel and left n-exact sequence dually.

Definition 2.15. Let $\mathcal{C}$ be an additive category. An n-exact sequence in $\mathcal{C}$ is a complex

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \tag{2.3}$$

in $Ch^n(\mathcal{C})$ such that $(d^0, ..., d^{n-1})$ is an n-ker of d^n, and $(d^1, ..., d^n)$ is an n-coker of d^0. The sequence (3.1) is called n-exact if it is both right n-exact and left n-exact.

Theorem 2.16. *Let A, B, $\{B_i | i \in I\}$, $\{A_j | j \in J, J is finite\}$ be modules over a ring R . Then there is isomorphisms of abelian groups:*

1. $Hom_R(A, \prod_{i\in I} B_i) \cong \prod_{i\in I} Hom_R(A, B_i)$.

2. $Hom_R(\oplus_{j\in J} A_j, B) \cong \oplus_{j\in J} Hom_R(A_j, B)$.

Theorem 2.17. *Let A, B, $\{B_i | i \in I\}$ be modules over a ring R. Then if I is finite there is isomorphisms of abelian groups: $Hom_R(A, \oplus_{i\in I} B_i) \cong \oplus_{i\in I} Hom_R(A, B_i)$.*

Proposition 2.18. *A direct product of R-modules $\prod_{i\in\mathbb{I}} J_i$ is injective if only if J_i is injective for every $i \in \mathbb{I}$.*

Corollary 2.19. *Let R be an integral domain and let K the field of fractions of R. Then K is an injective R-module.*

Corollary 2.20. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules. If Λ is finite and M_λ is injective for every $\lambda \in \Lambda$, then $\oplus_{\lambda\in\Lambda} M_\lambda$ is also injective.*

Theorem 2.21. *Let M be an R-module. Then M is injective if and only if for every short exact sequence $0 \longrightarrow A \xrightarrow{\theta} B \xrightarrow{\psi} C \longrightarrow 0$ of R-modules,*

$$0 \longrightarrow Hom_R(C, M) \xrightarrow{\Psi} Hom_R(B, M) \xrightarrow{\Theta} Hom_R(A, M) \longrightarrow 0$$

is also a short exact sequence, where $\Psi(f) = f\psi$ and $\Theta(f) = f\theta$.

Proposition 2.22. *Let R be a ring. A direct sum of R-modules $\sum_{i\in\mathbb{I}} P_i$ is projective if only if each P_i is projective.*

Proposition 2.23. *Every free left R-module is projective.*

Theorem 2.24. *A left R-module P is projective if and only if P is a direct summand of a free left R-module.*

Corollary 2.25.

1. *Every direct summand of a projective module is itself projective.*
2. *Every direct sum of projective modules is projective.*

Lemma 2.26. *Let R be a ring with identity. A unitary R-module J is injective if and only if for every left ideal L of R, any R-module homomorphism $L \longrightarrow J$ may be extended to an R-module homomorphism $R \longrightarrow J$:*

Example 2.27.

1. *Q is an injective Z-module by Lemma (2.26) since for every Z-homomorphism $f : nZ \longrightarrow Q$, where nZ is an ideal of Z for $0 \neq n \in Z$, there exists a Z-homomorphism $g : Z \longrightarrow Q$ defined by $g(z) = \frac{zf(n)}{n}$, so $g(nz) = \frac{(nz)f(n)}{n} = zf(n) = f(nz)$ for every $nz \in Z$.*

2. *Note that Z is not an injective Z-module since using the Z-homomorphism $f : 2Z \longrightarrow Z$ given by $f(2z) = z$, there is no Z-homomorphism $g : Z \longrightarrow Z$ such that $g(2z) = f(2z)$ for every $2z \in 2Z$. Otherwise, $1 = f(2) = g(2) = 2g(1)$, implying that $g(1) = \frac{1}{2}$. However, since $g(1) \in Z$, this is impossible.*

3 n-injective Module

Definition 3.1. Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n$, and d^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. An R-module M is n-injective if the sequence of R-module in $\mathcal{C}$ is left n-exact

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \ldots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

if there is $M \in \mathcal{C}$ the induced sequence of abelian groups

$$0 \longrightarrow Hom_{\mathcal{C}}(X^{n+1}, M) \xrightarrow{\hat{d^n}} Hom_{\mathcal{C}}(X^n, M) \xrightarrow{\hat{d^{n-1}}}$$

$$\ldots \xrightarrow{\hat{d^1}} Hom_{\mathcal{C}}(X^1, M) \xrightarrow{\hat{d^0}} Hom_{\mathcal{C}}(X^0, M)$$

is right n-exact.

Proposition 3.2. *Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n$, and d^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. A direct product of R-modules $\prod_{i\in\mathbb{I}} J_i$ is n-injective if only if J_i is n-injective for every $i \in \mathbb{I}$.*

Proof. Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n$, and d^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. The sequence of R-module in $\mathcal{C}$

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \ldots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

is left n-exact.

Suppose that $\prod_{i\in\mathbb{I}} J_i$ is n-injective. To show that, J_i is n-injective for each $i \in \mathbb{I}$. Now if there is $\prod_{i\in\mathbb{I}} J_i$ the induced sequence of abelian groups this sequence is

$$0 \longrightarrow Hom_{\mathcal{C}}(X^{n+1}, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^n}} Hom_{\mathcal{C}}(X^n, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^{n-1}}} Hom_{\mathcal{C}}(X^{n-1}, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^{n-2}}}$$

$$... \xrightarrow{\hat{d^2}} Hom_{\mathcal{C}}(X^2, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^1}} Hom_{\mathcal{C}}(X^1, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^0}} Hom_{\mathcal{C}}(X^0, \prod_{i\in\mathbb{I}} J_i)$$

is right n-exact. By Theorem 2.16, (1),

$$Hom_{\mathcal{C}}(X^i, \prod_{i\in\mathbb{I}} J_i) \cong \prod_{i\in I} Hom_R(X^i, J_i)$$

for each $i \in \mathbb{I}$. Then this sequence

$$0 \longrightarrow \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^{n+1}, J_i) \xrightarrow{\hat{d^n}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^n, J_i) \xrightarrow{\hat{d^{n-1}}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^{n-1}, J_i) \xrightarrow{\hat{d^{n-2}}}$$

$$... \xrightarrow{\hat{d^1}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^1, J_i) \xrightarrow{\hat{d^2}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^2, J_i) \xrightarrow{\hat{d^0}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^0, J_i)$$

is right n-exact. Then J_i is n-injective for each $i \in \mathbb{I}$.

Conversely, suppose that J_i is n-injective. To show that, $\prod_{i\in\mathbb{I}} J_i$ is n-injective for each $i \in \mathbb{I}$. Now if there is J_i the induced sequence of abelian groups this sequence is

$$0 \longrightarrow \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^{n+1}, J_i) \xrightarrow{\hat{d^n}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^n, J_i) \xrightarrow{\hat{d^{n-1}}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^{n-1}, J_i) \xrightarrow{\hat{d^{n-2}}}$$

$$... \xrightarrow{\hat{d^1}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^1, J_i) \xrightarrow{\hat{d^2}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^2, J_i) \xrightarrow{\hat{d^0}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^0, J_i)$$

is right n-exact. By Theorem 2.16, (1). Then this sequence

$$0 \longrightarrow Hom_{\mathcal{C}}(X^{n+1}, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^n}} Hom_{\mathcal{C}}(X^n, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^{n-1}}} Hom_{\mathcal{C}}(X^{n-1}, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^{n-2}}}$$

$$... \xrightarrow{\hat{d^2}} Hom_{\mathcal{C}}(X^2, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^1}} Hom_{\mathcal{C}}(X^1, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^0}} Hom_{\mathcal{C}}(X^0, \prod_{i\in\mathbb{I}} J_i)$$

is right n-exact. Then $\prod_{i\in\mathbb{I}} J_i$ is also n-injective. □

Corollary 3.3. *Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n$, and d^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. Let R be an integral domain and let K the field of fractions of R. Then K is an n-injective R-module.*

Proof. By Corollary 2.19, k is injective R-module. Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n$, and d^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. The sequence of R-module in $\mathcal{C}$

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

is left n-exact. By Theorem 2.21

$$0 \longrightarrow Hom_{\mathcal{C}}(X^{n+1}, K) \xrightarrow{\hat{d^n}} Hom_{\mathcal{C}}(X^n, M) \xrightarrow{\hat{d^{n-1}}}$$

$$... \xrightarrow{\hat{d^1}} Hom_{\mathcal{C}}(X^1, K) \xrightarrow{\hat{d^0}} Hom_{\mathcal{C}}(X^0, K)$$

is right n-exact. Then K is n-injective module.

□

Corollary 3.4. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules. If Λ is finite and M_λ is n-injective for every $\lambda \in \Lambda$, then $\oplus_{\lambda\in\Lambda} M_\lambda$ is also n-injective.*

Proof. Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n$, and d^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. The sequence of R-module in $\mathcal{C}$

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

is left n-exact.

Suppose that M_λ is n-injective. To show that, $\oplus_{\lambda\in\Lambda} M_\lambda$ is n-injective for each $\lambda \in \Lambda$. Now if there is M_λ the induced sequence of abelian groups this sequence is

$$0 \longrightarrow \oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^{n+1}, M_\lambda) \xrightarrow{\hat{d^n}} \oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^n, M_\lambda) \xrightarrow{\hat{d^{n-1}}}$$

$$\oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^{n-1}, M_\lambda) \xrightarrow{\hat{d^{n-2}}} ... \xrightarrow{\hat{d^2}} \oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^2, M_\lambda)$$

$$\xrightarrow{\hat{d^1}} \oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^1, M_\lambda) \xrightarrow{\hat{d^0}} \oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^0, M_\lambda)$$

is right n-exact. If Λ is finite by Theorem 2.17,

$$Hom_{\mathcal{C}}(X^i, \oplus_{\lambda\in\Lambda} M_\lambda) \cong \oplus_{\lambda\in\Lambda} Hom_R(X^i, M_\lambda)$$

for every $\lambda \in \Lambda$, Then this sequence

$$0 \longrightarrow Hom_{\mathcal{C}}(X^{n+1}, \oplus_{\lambda\in\Lambda}M_\lambda) \xrightarrow{\hat{d^n}} Hom_{\mathcal{C}}(X^n, \oplus_{\lambda\in\Lambda}M_\lambda)$$

$$\xrightarrow{\hat{d^{n-1}}} Hom_{\mathcal{C}}(X^{n-1}, \oplus_{\lambda\in\Lambda}M_\lambda) \xrightarrow{\hat{d^{n-2}}} ... \xrightarrow{\hat{d^2}} Hom_{\mathcal{C}}(X^2, \oplus_{\lambda\in\Lambda}M_\lambda)$$

$$\xrightarrow{\hat{d^1}} Hom_{\mathcal{C}}(X^1, \oplus_{\lambda\in\Lambda}M_\lambda) \xrightarrow{\hat{d^0}} Hom_{\mathcal{C}}(X^0, \oplus_{\lambda\in\Lambda}M_\lambda)$$

is right n-exact. Then $\oplus_{\lambda\in\Lambda}M_\lambda$ is also n-injective. □

Proposition 3.5. *Every R-module injective is not n-injective.*

Proof. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules. If $\oplus_{\lambda\in\Lambda}M_\lambda$ is injective, then M_λ is injective for every $\lambda \in \Lambda$ but $\oplus_{\lambda\in\Lambda}M_\lambda$ is not n-injective for every $\lambda \in \Lambda$ and then, M_λ is n-injective for every $\lambda \in \Lambda$. □

Definition 3.6. Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n$ is a morphism in $\mathcal{C}$. An R-module P is n-projective if the sequence of R-module in $\mathcal{C}$ is rightt n-exact

$$Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} Y^2 \xrightarrow{f^2} ... \xrightarrow{f^{n-1}} Y^n \xrightarrow{f^n} Y^{n+1}$$

if there is $P \in \mathcal{C}$ the induced sequence of abelian groups

$$Hom_{\mathcal{C}}(P, Y^0) \xrightarrow{\hat{f^0}} Hom_{\mathcal{C}}(P, Y^1) \xrightarrow{\hat{f^1}} Hom_{\mathcal{C}}(P, Y^2) \xrightarrow{\hat{f^2}}$$

$$... \xrightarrow{\hat{f^{n-2}}} Hom_{\mathcal{C}}(P, Y^{n-1}) \xrightarrow{\hat{f^{n-1}}} Hom_{\mathcal{C}}(P, Y^n) \xrightarrow{\hat{f^n}} Hom_{\mathcal{C}}(P, Y^{n+1}) \longrightarrow 0$$

is left n-exact.

Proposition 3.7. *Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n$ is a morphism in $\mathcal{C}$. A direct sum of R-modules $\oplus_{i\in\mathbb{I}}P_i$ is n-projective if only if P_i is n-projective for every $i \in \mathbb{I}$ and $\mathbb{I}$ is finite.*

Proof. Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. The sequence of R-module in $\mathcal{C}$

$$Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} Y^2 \xrightarrow{f^2} ... \xrightarrow{f^{n-1}} Y^n \xrightarrow{f^n} Y^{n+1}$$

is right n-exact.

Suppose that $\oplus_{i\in\mathbb{I}}P_i$ is n-projective. To show that, P_i is n-projective for each $i\in\mathbb{I}$. Now if there is $\oplus_{i\in\mathbb{I}}P_i$ the induced sequence of abelian groups this sequence is

$$Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^0)\xrightarrow{\hat{f^0}}Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^1)\xrightarrow{\hat{f^1}}Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^2)\xrightarrow{\hat{f^2}}$$
$$...\xrightarrow{\hat{f^{n-1}}}Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^n)\xrightarrow{\hat{f^n}}Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^{n+1})\longrightarrow 0$$

is left n-exact. If $\mathbb{I}$ is finite by Theorem 2.16, (2)

$$Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^j)\cong\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i,Y^j)$$

for every $i\in\mathbb{I}$, Then this sequence

$$\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i,Y^0)\xrightarrow{\hat{f^0}}\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i,Y^1)\xrightarrow{\hat{f^1}}Hom_{\mathcal{C}}(P_i,Y^2)\xrightarrow{\hat{f^2}}$$
$$...\xrightarrow{\hat{f^{n-1}}}\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i,Y^n)\xrightarrow{\hat{f^n}}\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i,Y^{n+1})\longrightarrow 0$$

is left n-exact. Then P_i is n-projective for each $i\in\mathbb{I}$.

Conversely, suppose that P_i is n-projective. To show that, $\oplus_{i\in\mathbb{I}}P_i$ is n-projective for every $i\in\mathbb{I}$ and $\mathbb{I}$ is finite. Now if there is P_i the induced sequence of abelian groups this sequence is

$$\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i,Y^0)\xrightarrow{\hat{f^0}}\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i,Y^1)\xrightarrow{\hat{f^1}}Hom_{\mathcal{C}}(P_i,Y^2)\xrightarrow{\hat{f^2}}$$
$$...\xrightarrow{\hat{f^{n-1}}}\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i,Y^n)\xrightarrow{\hat{f^n}}\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i,Y^{n+1})\longrightarrow 0$$

is left n-exact. If $\mathbb{I}$ is finite by Theorem 2.16, (2)

$$Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^j)\cong\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i,Y^j)$$

for every $i\in\mathbb{I}$. Then this sequence

$$Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^0)\xrightarrow{\hat{f^0}}Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^1)\xrightarrow{\hat{f^1}}Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^2)\xrightarrow{\hat{f^2}}$$
$$...\xrightarrow{\hat{f^{n-1}}}Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^n)\xrightarrow{\hat{f^n}}Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i,Y^{n+1})\longrightarrow 0$$

is left n-exact. Then $\oplus_{i\in\mathbb{I}}P_i$ is also n-projective. $\square$

4 One Open Problem

Using the following definitions, can we prove the following theorems about n-projective module and free R-module.

Proposition 4.1. *Every free left R-module is n-projective.*

Proposition 4.2.

1. *Every finite direct summand of a n-projective module is itself n-projective.*
2. *Every finite direct sum of n-projective modules is n-projective.*

Definition 4.3. Let n be a positive integer. An n-abelian category is an additive category $\mathcal{C}$ which satisfies the following axioms;

$(A0)$ The category $\mathcal{C}$ is idempotent complete.

$(A1)$ Every morphism in $\mathcal{C}$ has n-ker and n-coker.

$(A2)$ for every monomorphism $f^0 : X^0 \longrightarrow X^1$ in $\mathcal{C}$ and, for every n-coker $(f^0, f^1, ..., f^{n-1})$ of f^0, the following sequence n-exact:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} ... \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

$(A2^{op})$ for every epimorphism $g^n : X^n \longrightarrow X^{n+1}$ in $\mathcal{C}$ and, for every n-ker $(g^0, g^1, ..., g^{n-1})$ of g^n, the following sequence n-exact:

$$X^0 \xrightarrow{g^0} X^1 \xrightarrow{g^1} ... \xrightarrow{g^{n-1}} X^n \xrightarrow{g^n} X^{n+1}.$$

Now one can investigate a divisible modules in n-additive abelian category. Next one can obtain all of the result of them as we obtained in this paper, and it is an open problem.

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Trifunction Bihemivariational Inequalities

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Abstract

In this paper, we consider a new class of hemivariational inequalities, which is called the trifunction bihemivariational inequality. We suggest and analyze some iterative methods for solving the trifunction bihemivariational inequality using the auxiliary principle technique. The convergence analysis of these iterative methods is also considered under some mild conditions. Several special cases are also considered. Results proved in this paper can be viewed as a refinement and improvement of the known results.

1 Introduction

Variational inequalities theory introduced in 1964 by Stampacchia [31] can be viewed as a novel and significant generalization of the variational principles. The origin of the variational principles can be traced back to Euler, Newton, Lagrange and Bernoulli's brothers. These variational principles have emerged as a powerful tool to investigate and study a wide class of unrelated problems arising in industrial, regional, physical, pure and applied sciences in a unified and general framework. Variational inequalities have been extended and generalized in several direction using novel and new techniques. Panagiotopoulos [28] introduced the hemivariational inequalities by using the concept of the generalized directional derivatives of nonconvex and nondifferentiable functions. This class has important

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applications in structural analysis and nonconvex optimization. It has been shown [7] that, if a nonsmooth and nonconvex superpotential of a structure is quasidifferentiable, then these problems can be studied in the general framework of hemivariational inequalities. The solution of the hemivariational inequalities gives the position of the state equilibrium of the structure. We would like to point out that the hemivariational inequalities include the problem of finding the difference of two monotone operators, which is itself an interesting problem, see [8, 28].

Noor and Oettli [16] introduced triequilibrium problems and have shown variational inequalities, fixed-point problems, Nash equilibrium problems and saddle-point problems can be studied in the framework of triequilibrium problems. Thus it is clear that hemivariational inequalities and equilibrium problems are different generalizations of variational inequalities. Noor and Noor [17] investigated the trifunction hemivariational inequalities, which can be viewed a significant extension of variational inequalities and hemivariational inequalities. We would like to emphasize that hemivariational inequality theory provides us with a simple, natural, unified, novel and general framework to study an extensive range of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, circuit analysis, transportation, oceanography, operations research, finance, economics, and optimization.

Convexity theory is a branch of mathematical sciences with a wide range of applications in industry, physics, social, regional and engineering sciences. The general theory of the convexity started soon after the introduction of differential and integral calculus by Newton and Leibnitz, although some individual optimization problems had been investigated before that. It is worth mentioning that variational inequalities represent the optimality conditions for the differentiable convex functions on the convex sets. The convex sets and convex functions have been extended and generalized in several directions using innovative ideas to consider completed problems. See an excellent book by Cristescu and Lupsa [3]. Inspired by the research work going on in this field, Noor and Noor [21, 22, 23, 24] introduced and and considered a new class of

nonconvex sets and nonconvex functions with respect to an arbitrary bifunction. This class of nonconvex set is called the biconvex set and the noncovex function is called biconvex function. functions is called the biconvex functions. Noor et al [19, 21, 22, 23, 24, 26, 27] have studied some basic properties of the biconvex functions. It have been shown that the biconvex functions have characterizations as the convex functions enjoy. In particular, it have been shown that the optimization conditions of the differentiable biconvex functions are characterized by a class of variational inequalities, called the bivariational inequalities, see [19, 21, 22, 23, 24, 26, 27] and references therein.

Variational inequalities and hemivariational inequalities have witnessed an explosive growth in theoretical advances, algorithmic developments and applications across almost all disciplines of engineering, pure and applied sciences. There are several methods for solving variational inequalities and bivariational inequalities. Due to the nature of the hemivariational inequalities, projection and resolvent methods can not be applied for solving hemivariational inequalities. In recent years, the auxiliary principle technique is being used to suggest and analyze some iterative methods for solving variational inequalities and equilibrium problems. Glowinski, Lions and Tremolieres [5] used this technique to study the existence problem for mixed variational inequalities, whereas Noor [8, 11, 12, 13, 14] and Zhu et al.[32] have used this approach to suggest and analyze some iterative methods for solving various classes of variational inequalities and equilibrium problems. In this paper, we again use the auxiliary principle technique to suggest several new iterative schemes for trifunction bihemivariational inequalities. We also prove that the convergence of these methods require either pseudomonotonicity or partially relaxed strongly monotonicity. These are weaker conditions than monotonicity. As a special case, we obtain new iterative schemes for solving bihemivariational inequalities, variational inequalities and optimization problem. The comparison of these methods with other methods is a subject of future research.

2 Preliminaries and Basic Results

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$ respectively. Let K be a nonempty set in H.

We now recall some concepts of biconvex sets and biconvex functions, which are mainly due to Noor et al. [21, 22, 23, 24].

Definition 2.1. The set K_β in H is said to be *biconvex set* with respect to an arbitrary bifunction $\beta(\cdot - \cdot)$, if

$$u + \lambda\beta(v-u) \in K_\beta, \qquad \forall u, v \in K_\beta, \lambda \in [0,1].$$

The biconvex set K_β is also called β-connected set. If $\beta(v-u) = v-u$, then the biconvex set K_β is a convex set, but the converse is not true. For example, the set $K_\beta = R - (-\frac{1}{2}, \frac{1}{2})$ is an biconvex set with respect to β, where

$$\beta(v-u) = \begin{cases} v-u, & \text{for} \quad v > 0, u > 0 \quad \text{or} \quad v < 0, u < 0 \\ u-v, & \text{for} \quad v < 0, u > 0 \quad \text{or} \quad v < 0, u < 0. \end{cases}$$

It is clear that K_β is not a convex set.

Remark 2.1. We would like to emphasize that, if $u + \beta(v-u) = v, \quad \forall u, v \in K_\beta$, then $\beta(v-u) = v-u$. Consequently, the β-biconvex set reduces to the convex set K. Thus, $K_\beta \subset K$. This implies that every convex set is a biconvex set, but the converse is not true.

Definition 2.2. The function F on the biconvex set K_β is said to be *strongly biconvex*, if

$$\begin{aligned} F(u + \lambda\beta(v-u)) \quad &\leq \quad (1-\lambda)F(u) + \lambda F(v) \\ &\quad -\nu\lambda(1-\lambda)\|\beta(v-u)\|^2, \quad \forall u, v \in K_\beta, \lambda \in [0,1]. \end{aligned}$$

Note that every convex function is a biconvex, but the converse is not true.

If $\lambda = \frac{1}{2}$, then the function F satisfies

$$F(\frac{2u + \beta(v-u)}{2}) \leq \frac{1}{2}\{F(u) + F(v)\} - \nu\frac{1}{4}\|\beta(v-u)\|^2, \quad \forall u, v \in K_\beta,$$

which is called Jensen biconvex function.

If $\nu = 0$, then Definition(2.2) reduces to

Definition 2.3. The function F on the biconvex set K_β is said to be *biconvex*, if

$$F(u+\lambda\beta(v-u)) \leq (1-\lambda)F(u) + \lambda F(v) \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

We now consider the biconvex function on the interval $I_\beta = [a, a+\beta(b-a)]$.

Definition 2.4. Let $I_\beta = [a, a+\beta(b-a)]$. Then F is a *biconvex function*, if and only if,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & a+\beta(b-a) \\ F(a) & F(x) & F(b) \end{vmatrix} \geq 0; \quad a \leq x \leq a+\beta(b-a).$$

One can easily show that the following are equivalent:

1. F is a biconvex function.
2. $F(x \leq F(a) + \frac{F(b)-F(a)}{\beta(b-a)}(x-a)$.
3. $\frac{F(x)-F(a)}{x-a} \leq \frac{F(b)-F(a)}{\beta(b-a)}$.
4. $\frac{F(a)}{(\beta(b-a)))(a-x)} + \frac{F(x)}{(x-a)-\beta(b-a))(a-x)} + \frac{F(b)}{\beta(b-a)(x-b)} \leq 0$,

where $x = a+\lambda\beta(b-a) \in [a, a+\beta(b-a)]$.

To derive the main results, we need the following assumption regarding the bifunction $\beta(\cdot - \cdot)$.

Condition M. The bifunction $\beta(,-,)$ is said to satisfy the following assumptions:

$$\begin{aligned} (i). \quad \beta(\gamma\beta(v-u)) &= \gamma\beta(v-u), \quad \forall u, v \in K_\beta, \quad \gamma \in R^n. \\ (ii). \quad \beta(v-u-\gamma\beta(v-u)) &= (1-\gamma)\beta(v-u), \quad \forall u, v \in K_\beta. \end{aligned}$$

Remark 2.2. Let $\beta(\cdot - \cdot) : K_\beta \times K_\beta \to H$ satisfy the assumption

$$\beta(v-u) = \beta(v-z) + \beta(z-u), \quad \forall u, v, z \in K_\beta.$$

One can easily show that $\beta(v-u)=0 \quad \Leftrightarrow \quad u=v, \quad \forall u,v \in K_\beta$. Consequently $\beta(v-u)=0$, for $v=u \in K_\beta$. Also $\beta(v-u)+\beta(u-v)=0, \quad \forall u,v,z \in K_\beta$. This implies that the bifunction $\beta(.-.)$ is skew symmetric.

Let $f: H \longrightarrow R$ be a locally Lipschitz continuous function. Let Ω be an open bounded subset of R^n. First of all, we recall the following concepts and results from nonsmooth analysis [2].

Definition 2.5. Let f be locally Lipschitz continuous at a given point $x \in H$ and v be any other vector in H. The *Clarke's generalized bidirectional derivative* of f at x in the direction $\beta(v-u)$, denoted by $f^0(x,\beta(v-u))$, is defined as

$$f^0(x,\beta(v-u)) = \lim_{t\to 0^+} \sup_{h\to 0} \frac{f(x+h+t\beta(v-u))-f(x+h)}{t}.$$

If $\beta(v-u)=v$, then Definition (2.5) reduces to the following concepts which are mainly due to Clarke [2].

Definition 2.6. *[2]* Let f be locally Lipschitz continuous at a given point $x \in H$ and v be any other vector in H. The *Clarke's generalized bidirectional derivative* of f at x in the direction v, denoted by $f^0(x,v)$, is defined as

$$f^0(x,v) = \lim_{t\to 0^+} \sup_{h\to 0} \frac{f(x+h+tv)-f(x+h)}{t}.$$

The generalized gradient of f at x, denoted $\partial f(x)$, is defined to be subdifferential of the function $f^0(x;v)$ at 0. That is

$$\partial f(x) = \{w \in H : \langle w,v\rangle \leq f^0(x;v), \quad \forall v \in H.\}.$$

If f is convex on K and locally Lipschitz continuous at $x \in K$, then $\partial f(x)$ coincides with the subdifferential $f'(x)$ of f at x in the sense of convex analysis , and $f^0(x;v)$ coincides with the directional derivative $f'(x;v)$ for each $v \in H$, that is, $f^0(x;v) = \langle f'(x),v\rangle, \quad \forall v \in H.$

For a given nonlinear trifunction $F(.,.,.): K_\beta \times K_\beta \times K_\beta \longrightarrow H$ and a nonlinear continuous operator $T: K_\beta \longrightarrow H$, consider the problem of finding $u \in K_\beta$ such

that

$$F(u, Tu, \beta(v-u)) + \int_\Omega f^0(u; \beta(v-u))d\Omega \geq 0, \quad \forall v \in K_\beta, \tag{2.1}$$

which is called the trifunction bihemivariational inequality.

Here $f^0(u;\beta(v-u)) := f^0(x, u; \beta(v-u)) := f^0(x, u(x); \beta(v(x)-u(x)))$ denotes the generalized bidirectional derivative of the function $f(x, .)$ at $u(x)$ in the direction $v(x) - u(x)$.

We now discuss some special cases of the trifunction bihemivariational inequalities (2.1).

(I). If $F(u, Tu, \beta(v-u)) = W(u, \beta(v-u))$, where $B(., .)$ is a continuous bifunction, then problem (2.1) is equivalent to finding $u \in K_\beta$ such that

$$W(u, \beta(v-u)) + \int_\Omega f^0(u; \beta(v-u))d\Omega \geq 0, \quad \forall v \in K_\beta, \tag{2.2}$$

which is called the bifunction bihemivariational inequality and appears to be a new one.

(II). If $F(u, Tu, \beta(v-u)) = \langle Au, \beta(v-u)\rangle$, where A is a nonlinear operator, then problem (2.1) is equivalent to finding $u \in K_\beta$ such that

$$\langle Au, \beta(v-u)\rangle + \int_\Omega f^0(u; \beta(v-u))d\Omega \geq 0, \quad \forall v \in K_\beta, \tag{2.3}$$

which is known as the bihemivariational inequality.

(III). If $F(u, Tu, \beta(v-u)) = \langle Au, v-u\rangle$, where A is a nonlinear operator, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Au, v-u\rangle + \int_\Omega f^0(u; v-u)d\Omega \geq 0, \quad \forall v \in K, \tag{2.4}$$

which is known as the hemivariational inequality introduced and studied by Panagiotopoulos [28, 29] in order to formulate variational principles connected to energy functions which are neither convex nor smooth. It is has been shown that the technique of hemivariational inequalities is very efficient to describe the behaviour of complex structure arising in engineering and industrial sciences.

(IV). If f is a differentiable convex function, then problem (2.1) is equivalent to finding $u \in K_\beta$ such that

$$F(u, Tu, \beta(v-u)) + \langle f'(u), \beta(v-u)\rangle \geq 0, \quad \forall v \in K_\beta, \tag{2.5}$$

which is known as the mildly nonlinear trifunction bihemivariational inequality and appear to be a new one.

(V). If $f = 0$, then problem (2.1) is equivalent to finding $u \in K_\beta$ such that

$$F(u, Tu, \beta(v-u)) \quad \geq 0, \quad \forall v \in K_\beta, \tag{2.6}$$

which is called the trifunction bivariational inequality.

In brief, for suitable and appropriate choice of the trifunction, one can obtain several classes of bihemivariational and bivariational inequalities. This clear shows that the problem (2.1) is more general and flexible and includes the previous ones as special cases.

Definition 2.7. The trifunction $F(.,.,.)$ and the operator T is said to be:
(a) *jointly bimonotone,* if

$$F(u, Tu, \beta(v-u)) + F(v, Tv, \beta(u-v)) \leq 0, \quad \forall u, v \in K_\beta.$$

(b) *jointly pseudo-bimonotone* with respect to $\int_\Omega f^0(u; \beta(v-u))d\Omega$, if

$$\begin{aligned} & F(u, Tu, \beta(v-u)) + \int_\Omega f^0(u; \beta(v-u))d\Omega \geq 0 \\ & \Longrightarrow \\ & -F(v, Tv, \beta(u-v)) - \int_\Omega f^0(u; \beta(v-u))d\Omega \geq 0, \quad \forall u, v \in K_\beta. \end{aligned}$$

(c) *partially relaxed strongly jointly bimonotone,* if there exists a constant $\gamma > 0$ such that

$$F(u, Tu, \beta(v-u)) + F(v, Tv, \beta(z-v)) \leq \gamma\|\beta(u-z)\|^2, \quad \forall u, v, z \in K_\beta.$$

Note that for $z = u$ partially relaxed strongly jointly bimonotonicity reduces to jointly bimonotonicity. This shows that partially relaxed strongly jointly bimonotonicity implies jointly bimonotonicity, but the converse is not true.

Definition 2.8. The function $\int_\Omega f^0(u;\beta(v-u))d\Omega$ is said to be *partially relaxed strongly bimonotone*, if there exists a constant $\alpha > 0$ such that

$$\int_\Omega f^0(u;\beta(v-u))d\Omega + \int_\Omega f^0(z;\beta(u-v))d\Omega \leq \alpha\|\beta(z-v)\|^2, \quad \forall u,v,z \in H.$$

Note that for $z = v$, partially relaxed strongly bimonotonicity reduces to relaxed strongly bimonotonicity.

3 Main Results

In this section, we suggest and analyze some iterative methods for solving trifunction bihemivariational inequality (2.1) using the auxiliary principle technique of Glowinski, Lions and Tremolieres [5] involving Bregman distance function as developed by Noor [11, 12, 13, 14, 15], Noor et al. [17, 18, 19, 20] and Zhu et al. [32].

For the readers convenience, we recall some basic properties of the Bregman convex functions [2]. For strongly convex functions f, we define the Bregman distance function as

$$B(v,u) = f(v) - f(u) - \langle f'(u), v-u\rangle \geq \alpha\|v-u\|^2, \quad \forall u,v \in K. \tag{3.1}$$

It is important to emphasize that various types of function f give different Bregman distance function. We give the following important examples of some practical important types of function f and their corresponding Bregman distance functions.

Examples

1. If $f(v) = \|v\|^2$, then $B(v,u) = \|v-u\|$, which is the squared Euclidean distance (SE).

2. If $f(v) = \sum_{i=1}^n a_i \log v_i$, which is known as Shannon entropy, then its corresponding Bregman distance is given as

$$B(v,u) = \sum_{i=1}^n \left(v_i \log(\frac{v_i}{u_i}) + u_i - v_i \right),$$

This distance is called Kullback-Leibler distance (KL) and has become a very important tool in several areas of applied mathematics such as machine learning.

3. If $f(v) = -\sum_{i=1}^{n} \log v_i$, which is called Burg entropy, then its corresponding Bregman distance is given as

$$B(v,u) = \sum_{i=1}^{n} \left(\log \frac{v_i}{u_i} + \frac{v_i}{u_i} - 1 \right).$$

This is called Itakura-Saito distance (IS), which is very important in the information theory, data analysis and machine learning.

Remark 3.1. It is a challenging problem to explore the applications of Bregman distance function for other types of nonconvex functions such as biconvex, k-convex functions, preinvex functions and harmonic functions.

For a given $u \in K_\beta$ satisfying (2.1), consider the auxiliary problem of finding $w \in K_\beta$ such that

$$\begin{aligned} \rho F(w, Tw, \beta(v-w)) \quad &+ \quad \langle E'(w) - E'(u), \beta(v-w) \rangle \\ &+ \quad \rho \int_\Omega f^0(w; \beta(v-w)) d\Omega \geq 0, \quad \forall v \in K_\beta, \end{aligned} \tag{3.2}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential biconvex function $E(u)$ at $u \in K_\beta$.

We note that, if $w = u$, then clearly w is solution of the problem (2.1). This observation enables us to suggest and analyze the following iterative method for solving (2.1).

Algorithm 3.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\rho F(u_{n+1}, Tu_{n+1}, \beta(v - u_{n+1})) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle \\ &+\rho \int_\Omega f^0(u_{n+1}; \beta(v - u_{n+1})) d\Omega \geq 0, \quad \forall v \in K_\beta. \end{aligned} \tag{3.3}$$

Algorithm 3.1 is called the proximal method for solving problem (2.1). In passing, we remark that the proximal point method was suggested by Martinet [6] in the context of convex programming problems as regularization technique. For the recent developments and applications of the proximal point algorithms, see [11, 12, 13, 14, 15, 19, 32] and the references therein.

If $F(u, Tu, \beta(v-u)) = W(u, \beta(v-u))$, then Algorithm 3.1 collapses to the following method for solving the bifunction bihemivariational inequality (2.2).

Algorithm 3.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\rho W(u_{n+1}, \beta(v-u_{n+1})) + \langle E'(u_{n+1}) - E'(u_n), \beta(v-u_{n+1})\rangle \\ +\rho \int_\Omega f^0(u_{n+1}; \beta(v-u_{n+1}))d\Omega \geq 0, \quad \forall v \in K_\beta,\end{aligned}$$

If $F(u, Tu, \beta(v-u)) = \langle Au, \beta(v-u)\rangle$, then Algorithm 3.1 reduces to:

Algorithm 3.3. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}&\langle \rho Au_{n+1} + E'(u_{n+1}) - E'(u_n), \beta(v-u_{n+1})\rangle \\ &+\rho \int_\Omega f^0(u_{n+1}; \beta(v-u_{n+1}))d\Omega \geq 0, \quad \forall v \in K_\beta,\end{aligned}$$

is called the proximal point method for solving bihemivariational inequalities (2.3) and appears to be a new one.

If $f(x, u) = 0$, then Algorithm 3.1 collapses to:

Algorithm 3.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}, Tu_{n+1}, \beta(v-u_{n+1})) + \langle E'(u_{n+1}) - E'(u_n), \beta(v-u_{n+1})\rangle \geq 0, \quad \forall v \in K_\beta.$$

In brief, for suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational-hemivariational inequalities and related problems.

We now study the convergence analysis of Algorithm 3.1, which is the main motivation of our next result.

Theorem 3.1. *Let $F(.,.,.)$ and the operator T be jointly pseudomonotone with respect to $\int_\Omega f^0(u;\beta(v-u))d\Omega$. Let E be differentiable strongly biconvex function with module $\mu > 0$. Then the approximate solution u_{n+1} obtained from Algorithm 3.1 converges to a solution $u \in K_\beta$ satisfying (2.1).*

Proof. Let $u \in K_\beta$ be a solution of (2.1). Then

$$F(u, Tu, \beta(v-u)) + \int_\Omega f^0(u;\beta(v-u))d\Omega \geq 0, \quad \forall v \in K_\beta,$$

implies that

$$-F(v, Tv, \beta(u-v)) - \int_\Omega f^0(x, u;\beta(v-u))d\Omega \geq 0, \quad \forall v \in K_\beta, \tag{3.4}$$

since $F(.,.,.)$ is jointly pseudomonotone with respect to $\int_\Omega f^0(u;\beta(v-u))d\Omega$.

Taking $v = u$ in (3.3) and $v = u_{n+1}$ in (3.4), we have

$$\begin{aligned} \rho F(u_{n+1}, Tu_{n+1}, \beta(u-u_{n+1})) &+ \langle E'(u_{n+1}) - E'(u_n), \beta(u-u_{n+1})\rangle \\ &\geq -\rho \int_\Omega f^0(u_{n+1};\beta(u-u_{n+1}))d\Omega. \end{aligned} \tag{3.5}$$

and

$$-F(u_{n+1}, Tu_{n+1}, \beta(u-u_{n+1})) - \int_\Omega f^0(u;\beta(u_{n+1}-u))d\Omega \geq 0. \tag{3.6}$$

We now consider the function Bregman distance function

$$\begin{aligned} B(u, w) &= E(u) - E(w) - \langle E'(w), \beta(u-w)\rangle \\ &\geq \mu\|\beta(u-w)\|^2, \quad \text{(using strongly biconvexity of } E). \end{aligned} \tag{3.7}$$

where $\mu > 0$ is a constant.

Now combining (3.7) and (3.4), we have

$$\begin{aligned} B(u,u_n)-B(u,u_{n+1}) &= E(u_{n+1})-E(u_n)-\langle E'(u_{n+1}),\beta(u_{n+1}-u_n)\rangle \\ &\quad +\langle E'(u_{n+1})-E'(u_n),\beta(u-u_{n+1})\rangle \\ &\geq \mu\|\beta(u_{n+1}-u_n)\|^2+\langle E'(u_{n+1})-E'(u_n),\beta(u-u_{n+1})\rangle \\ &\geq \mu\|\beta(u_{n+1}-u_n)\|^2-\rho F(u_{n+1},Tu_{n+1},\beta(u-u_{n+1})) \\ &\quad -\rho\int_\Omega f^0(u_n;\beta(u-u_{n+1}))d\Omega \\ &\geq \mu\|\beta(u_{n+1}-u_n)\|^2, \end{aligned}$$

where we have used (3.6).

If $u_{n+1}=u_n$, then clearly u_n is a solution of the trifunction bihemivariational inequality (2.1). Otherwise, it follows that $B(u,u_n)-B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty}\|\beta(u_{n+1}-u_n)\|=0\Rightarrow\lim_{n\to\infty}u_{n+1}=u.$$

Now using the technique of Zhu and Marcotte [20], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the trifunction bihemivariational inequality (2.1). □

It is well-known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving (2.1) using the auxiliary principle technique.

For a given $u\in K_\beta$ satisfying (2.1), find $w\in K_\beta$ such that

$$\begin{aligned} \rho F(u,Tu,\beta(v-w)) &+ \langle E'(w)-E'(u),\beta(v-w)\rangle \\ &+ \rho\int_\Omega f^0(u;\beta(v-w))d\Omega,\quad \forall v\in K_\beta, \end{aligned} \tag{3.8}$$

where $E'(u)$ is the differential of a strongly biconvex function $E(u)$ at $u\in K_\beta$.

Note that problems (3.2) and (3.8) are quite different problems.It is clear that for $w = u$, w is a solution of (2.1). This fact allows us to suggest and analyze another iterative method for solving trifunction bihemivariational inequality (2.1).

Algorithm 3.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho F(w_n, Tw_n, \beta(v - u_{n+1})) &+ \langle E'(u_{n+1}) - E'(w_n), \beta(v - u_{n+1})\rangle \\ &\geq -\rho \int_\Omega (w_n; \beta(v - u_{n+1}))d\Omega, \quad \forall v \in K_\beta, \quad (3.9) \\ \mu F(u_n, Tu_n, \beta(v - w_n)) &+ \langle E'(w_n) - E'(u_n), \beta(v - w_n)\rangle \\ &\geq -\mu \int_\Omega (u_n; \beta(v - w_n))d\Omega, \quad \forall v \in K_\beta. \quad (3.10) \end{aligned}$$

Note that for $F(u, Tu, \beta(v - u)) = W(u, \beta(v - u))$, Algorithm 3.5 reduces to:

Algorithm 3.6. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho W(w_n, \beta(v - u_{n+1})) &+ \langle E'(u_{n+1}) - E'(w_n), \beta(v - u_{n+1})\rangle \\ &\geq -\rho \int_\Omega (w_n; \beta(v - u_{n+1}))d\Omega, \quad \forall v \in K_\beta, \\ \mu W(u_n, \beta(v - w_n)) &+ \langle E'(w_n) - E'(u_n), \beta(v - w_n)\rangle \\ &\geq -\mu \int_\Omega (u_n; \beta(v - w_n))d\Omega, \quad \forall v \in K_\beta, \end{aligned}$$

which is called the predictor-corrector method for solving the bifunction bihemivariational inequality (2.3).

For $F(u, Tu, \beta(v-u)) = \langle Au, \beta(v-u)\rangle$ Algorithm 3.5 collapses to the method for solving the bivariational inequalities (2.2).

Algorithm 3.7. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Aw_n + E'(u_{n+1}) - E'(w_n), \beta(v - u_{n+1})\rangle \geq -\rho \int_\Omega (w_n; \beta(v - u_{n+1}))d\Omega, \forall v \in K_\beta,$$

$$\langle \mu Au_n + E'(w_n) - E'(u_n), \beta(v - w_n)\rangle \geq -\mu \int_\Omega (u_n; \beta(v - w_n))d\Omega, \quad \forall v \in K_\beta,$$

which is called the predictor-corrector method for solving the bihemivariational inequalities (2.2).

If $f(.;.) = 0$, then Algorithm 3.5 reduces to the following iterative method for solving trifunction bivaraiational inequalities (2.5).

Algorithm 3.8. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(w_n, Tw_n, \beta(v - u_{n+1})) + \langle E'(u_{n+1}) - E'(w_n), \beta(v - u_{n+1})\rangle \geq 0, \quad \forall v \in K_\beta,$$

$$\mu F(u_n, Tu_n, \beta(v - w_n)) + \langle E'(w_n) - E'(u_n), \beta(v - w_n)\rangle \geq 0, \quad \forall v \in K_\beta.$$

Similarly for suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving hemivariational and variational inequalities.

We now consider the convergence analysis of Algorithm 3.5 using essentially the technique of Theorem 3.1. For the sake of completeness and to convey an idea of the technique, we sketch the main points.

Theorem 3.2. *Let $F(.,.,.)$ and the operator T be partially relaxed strongly jointly bimonotone with a constant $\gamma > 0$ and let $\int_\Omega f^0(u; v - u)d\Omega$ be partially relaxed strongly bimonotone with a constant $\alpha > 0$. If E is strongly biconvex function with modulus $\beta > 0$ and $0 < \rho < \beta/(\alpha + \gamma)$, $0 < \mu < \beta/(\alpha + \gamma)$, then the approximate solution u_{n+1} obtained from Algorithm 3.5 converges to a solution $u \in K_\beta$ of (2.1).*

Proof. Let $u \in K_\beta$ be solution of (2.1). Then

$$\rho\{F(u, Tu, \beta(v - u)) + \int_\Omega f^0(u; \beta(v - u))d\Omega\} \geq 0, \quad \forall v \in K_\beta \tag{3.11}$$

$$\mu\{F(u, Tu, \beta(v - u)) + \int_\Omega f^0(u; \beta(v - u))d\Omega\} \geq 0, \quad \forall v \in K_\beta, \tag{3.12}$$

where $\rho > 0$ and $\mu > 0$ are constants.

Setting $v = u_{n+1}$ in (3.11) and $v = u$ in (3.9), we have

$$\rho\{F(u, Tu, \beta(u_{n+1} - u)) + \int_\Omega f^0(x, u; \beta(u_{n+1} - u))d\Omega\} \geq 0. \tag{3.13}$$

and

$$\begin{aligned}\rho F(w_n, Tw_n, \beta(u-u_{n+1})) &+ \langle E'(u_{n+1}) - E'(w_n), \beta(u-u_{n+1})\rangle \\ &\geq -\rho \int_\Omega f^0(x, w_n; \beta(u-u_{n+1}))d\Omega. \quad (3.14)\end{aligned}$$

As in Theorem 3.1 and from (3.13) and (3.13), we have

$$\begin{aligned} & B(u,w_n) - B(u,u_{n+1}) \\ = \;& E(u_{n+1}) - E(w_n) - \langle E'(u_{n+1}), \beta(u_{n+1}-w_n)\rangle \\ & +\langle E'(u_{n+1}) - E'(w_n), \beta(u-u_{n+1})\rangle \\ \geq \;& \mu\|\beta(u_{n+1}-w_n)\|^2 + \langle E'(u_{n+1}) - E'(w_n), \beta(u-u_{n+1})\rangle \\ \geq \;& \mu\|\beta(u_{n+1}-w_n)\|^2 - \rho F(w_n, Tw_n, \beta(u-u_{n+1})) \\ & -\rho\int_\Omega f^0(w_n; \beta(u-u_{n+1}))d\Omega \\ \geq \;& \mu\|\beta(u_{n+1}-w_n)\|^2 \\ & -\rho\{F(w_n, Tw_n, \beta(u-u_{n+1})) + F(u, Tu, \beta(u_{n+1}-u))\} \\ & -\rho\{\int_\Omega f^0(u; \beta(u_{n+1}-u))d\Omega + \int_\Omega f^0(w_n; \beta(u-u_{n+1}))d\Omega\} \\ \geq \;& \mu\|\beta(u_{n+1}-w_n)\|^2 - \rho(\alpha+\gamma)\|\beta(u_{n+1}-w_n)\|^2 \\ = \;& \{\mu - \rho(\alpha+\gamma)\}\|\beta(u_{n+1}-w_n)\|^2,\end{aligned}$$

where we have used the fact that $F(.,.,.)$ and $\int_\Omega f^0(.;.)d\Omega$ are partially relaxed strongly bimonotone with constants $\alpha > 0$ and $\gamma > 0$ respectively.

In a similar way, we can obtain

$$B(u,u_n) - B(u,w_n) \geq \{\beta - \mu(\alpha+\gamma)\}\|\beta(w_n-u_n)\|^2.$$

If $u_{n+1} = w_n = u_n$, then clearly u_n is a solution of the trifunction hemivariational inequality (2.1). Otherwise, for $0 < \rho < \frac{\beta}{\alpha+\gamma}$ and $0 < \mu < \frac{\beta}{\alpha+\gamma}$, it follows that the sequences $B(u,w_n) - B(u,u_{n+1})$ and $B(u,u_n) - B(u,w_n)$ are nonnegative and we must have

$$\lim_{n\to\infty} \|\beta(u_{n+1}-w_n)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\beta(w_n-u_n)\| = 0.$$

Thus

$$\lim_{n\to\infty} \|\beta(u_{n+1}-u_n)\| = \lim_{n\to\infty} \|\beta(u_{n+1}-w_n)\| + \lim_{n\to\infty} \|\beta(w_n-u_n)\| = 0$$

Now using the technique of Zhu and Marcotte [20], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the trifunction bihemivariational inequality (2.1). □

We now suggest and analyze some new iterative methods for solving the trifunction bihemivariational inequality (2.1) using the auxiliary principle technique of Glowinski, Lions and Tremolieres [10] without the Bregman distance function as developed by Noor [16-24].

For a given $u \in K_\beta$ satisfying (2.1), find $w \in K_\beta$ such that

$$\begin{aligned} \rho F(u, Tu, \beta(v-w)) &+ \langle w-u, v-w\rangle \\ &+ \rho \int_\Omega f^0(u;\beta(v-w))d\Omega \geq 0, \quad \forall v \in K_\beta, \quad (3.15) \end{aligned}$$

where $\rho > 0$ is a constant. Problem (3.15) is known as the auxiliary trifunction bihemivariational inequality. We note that if $w = u$, then clearly w is a solution of the (2.1). This observation enables us to suggest and analyze the following iterative method for solving (2.1).

Algorithm 3.9. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho F(w_n, Tw_n, \beta(v-w_n)) &+ \langle u_{n+1}-w_n, v-u_{n+1}\rangle \\ &+ \rho \int_\Omega f^0(u;\beta(v-u_{n+1}))d\Omega \geq 0, \quad \forall v \in K_\beta \quad (3.16) \\ \eta F(u_n, Tu_n, v-u_n) &+ \langle w_n-u_n, v-w_n\rangle \\ &+ \eta \int_\Omega f^0(u;\beta(v-w_n))d\Omega \geq 0, \quad \forall v \in K_\beta, \quad (3.17) \end{aligned}$$

where $\rho > 0$ and $\eta > 0$ are constants. Algorithm 3.9 is called the predictor-corrector method for solving the trifunction bihemivariational inequality (2.1).

We now study the convergence analysis of Algorithm 3.9.

Theorem 3.3. *Let $\bar{u} \in K_\beta$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.9 . If $F(.,.)$ is partially relaxed strongly monotone with a constant $\alpha > 0$ and the operator $\int_\Omega f^0(u;-,)d\Omega$ is partially relaxed strongly monotone with a constant $\gamma > 0$, then*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|w_n - \bar{u}\|^2 - (1 - 2\rho(\alpha + \gamma))\|u_{n+1} - w_n\|^2 \tag{3.18}$$

$$\|w_n - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\beta(\alpha + \gamma))\|w_n - u_n\|^2. \tag{3.19}$$

Proof. Let $\bar{u} \in K_\beta$ be a solution of (2.1). Then

$$\rho F(\bar{u}, T\bar{u}, \beta(v - \bar{u}) \quad + \quad \rho \int_\Omega f^0(\bar{u}; \beta(v - \bar{u}))d\Omega \geq 0, \quad \forall v \in K_\beta \tag{3.20}$$

$$\eta F(\bar{u}, T\bar{u}, \beta(v - \bar{u}) \quad + \quad \eta \int_\Omega f^0(\bar{u}; \beta(v - \bar{u}))d\Omega \geq 0, \quad \forall v \in K_\beta, \tag{3.21}$$

where $\rho > 0$ and $\eta > 0$ are constants.

Now taking $v = u_{n+1}$ in (3.20) and $v = \bar{u}$ in (3.16), we have

$$\rho F(\bar{u}, T\bar{u}, u_{n+1} - \bar{u}) + \rho \int_\Omega f^0(u; \beta(u_{n+1} - u))d\Omega \geq 0 \tag{3.22}$$

$$\begin{aligned} &\rho F(w_n, Tw_n, \bar{u} - w_n) + \langle u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle \\ &+\rho \int_\Omega f^0(u; \beta(\bar{u} - u_{n+1}))d\Omega \geq 0. \end{aligned} \tag{3.23}$$

Adding (3.22) and (3.23), we have

$$\begin{aligned} &\langle u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle \\ \geq\ & -\rho\{F(w_n, Tw_n, \beta(\bar{u} - w_n)) + F(\bar{u}, T\bar{u}, \beta(u_{n+1} - \bar{u}))\} \\ & -\rho\{\int_\Omega f^0(u; \beta(u_{n+1} - \bar{u}))d\Omega + \int_\Omega f^0(u; \beta(\bar{u} - u_{n+1}))d\Omega\} \\ \geq\ & -(\alpha + \gamma)\rho\|u_{n+1} - w_n\|^2, \end{aligned} \tag{3.24}$$

where we have used the fact that $F(.,.,.)$ is relaxed strongly monotone with constants $\alpha > 0$.

Recall the following result,

$$2\langle u, v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2, \quad \forall a, b \in H, \tag{3.25}$$

Setting $u = \bar{u} - u_{n+1}$ and $v = u_{n+1} - w_n$ in (3.25), (3.24) can be written as

$$\|u_{n+1} - \bar{u}\|^2 \leq \|\bar{u} - w_n\|^2 - (1 - 2(\alpha + \gamma)\rho)\|u_{n+1} - w_n\|^2,$$

the required (3.18).

Taking $v = \bar{u}$ in (3.21) and $v = w_n$ in (3.17), we obtain

$$\eta F(\bar{u}, T\bar{u}, \beta(w_n - \bar{u})) + \eta \int_\Omega f^0(u; \beta(w_n - \bar{u}))d\Omega \geq 0 \tag{3.26}$$

$$\eta F(u_n, Tu_n, \beta(\bar{u} - u_n)) + \langle w_n - u_n, \bar{u} - w_n \rangle + \eta \int_\Omega f^0(u_n; \beta(\bar{u} - w_n))d\Omega \geq 0. \tag{3.27}$$

Adding (3.26), (3.27) and rearranging the terms, we have

$$\langle w_n - u_n, \bar{u} - w_n \rangle \geq -\beta(\alpha + \gamma)\|u_n - w_n\|^2, \tag{3.28}$$

since $F(.,.,.)$ and $\int_\Omega f^0(u; -))d\Omega$ are partially relaxed strongly monotone with constants $\alpha > 0$ and $\gamma > 0$ respectively.

Now taking $v = w_n - u_n$ and $u = \bar{u} - w_n$ in (3.25), (3.28) can be written as

$$\|w_n - \bar{u}\|^2 \leq \|\bar{u} - u_n\|^2 - (1 - 2(\alpha + \gamma)\beta)\|w_n - u_n\|^2,$$

the required (3.19). □

Theorem 3.4. *Let H be a finite dimensional space and let $0 < \rho < 1/2(\alpha + \gamma)$, $0 < \beta < 1/2(\alpha + \gamma)$. If $\bar{u} \in K_\beta$ is a solution of (1) and u_{n+1} is an approximate solution obtained from Algorithm 3.10, then*

$$\lim_{n \longrightarrow \infty} (u_n) = \bar{u}.$$

Proof. Let $\bar{u} \in K_\beta$ be a solution of (2.1). Since $0 < \rho < 1/2(\alpha + \gamma)$ and $0 < \beta < 1/2(\alpha + \gamma)$, it follows from (3.18)and (3.19) that the sequences $\{\|w_n - \bar{u}\|\}$ and $\{\|\bar{u} - u_n\|\}$ are nonincreasing and consequently $\{u_n\}$ and $\{w_n\}$ are bounded.

Furthermore, we have

$$\sum_{n=0}^{\infty}(1-2(\alpha+\gamma)\rho)\|u_{n+1}-w_n\|^2 \leq \|w_0-\bar{u}\|^2$$
$$\sum_{n=0}^{\infty}(1-2(\alpha+\gamma)\beta)\|w_n-u_n\|^2 \leq \|u_0-\bar{u}|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-w_n\|=0 \quad \text{and} \quad \lim_{n\longrightarrow\infty}\|w_n-u_n\|=0.$$

Thus

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\| \leq \lim_{n\longrightarrow\infty}\|u_{n+1}-w_n\| + \lim_{n\longrightarrow\infty}\|w_n-u_n\| = 0. \tag{3.29}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (3.15), (3.16) and taking the limit $n_j \longrightarrow \infty$ and using (3.29), we have

$$F(\hat{u}, T\hat{u}, v-\hat{u}) + \int_{\Omega} f^0(\hat{u}; \beta(v-\hat{u}))d\Omega \geq 0, \quad \forall v \in K,$$

which implies that $\hat{u}$ solves the trifunction bihemivariational inequality (2.1) and

$$\|u_{n+1}-\hat{u}\|^2 \leq \|u_n-\hat{u}\|^2.$$

Thus, it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\longrightarrow\infty}(u_n) = \hat{u},$$

the required result. □

In recent years, inertial proximal methods [1] have been suggested and analyzed for maximal monotone operators associated with the discretizations of the differential equations in times, whereas Noor [12] has used the auxiliary principle technique to suggest an inertial method for variational inequalities, the converges of which requires only pseudomonotonicity, which is a weaker

condition than monotonicity. This clearly improves the convergence criteria of the inertial proximal method. We again use the auxiliary principle to suggest and analyze an inertial proximal method for solving the trifunction bihemivariational inequality(2.1).

For a given $u \in K_\beta$ satisfying (2.1), consider the problem of finding $w \in K_\beta$ such that

$$\begin{aligned} &\rho F(w, Tw, \beta(v-w)) + \langle w - u - \alpha(u-u), v - w\rangle \\ &\quad + \rho \int_\Omega f^0(u; \beta(v-w))d\Omega \geq 0, \quad \forall v \in K_\beta, \end{aligned} \tag{3.30}$$

where $\rho > 0$ and $\alpha > 0$ are constants.

It is clear that, if $w = u$, then u is a solution of (2.1). This fact allows us to suggest and analyze an iterative method for solving the trifunction bihemivariational inequality (2.1).

Algorithm 3.10. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\rho F(u_{n+1}, Tu_{n+1}, \beta(v - u_{n+1})) \\ &+\langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), v - u_{n+1}\rangle \\ &+\rho \int_\Omega f^0(u; \beta(v - u_{n+1}))d\Omega \geq 0, \quad \forall v \in K_\beta. \end{aligned} \tag{3.31}$$

For $\alpha_n = 0$, Algorithm 3.11 reduces to :

Algorithm 3.11. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\rho F(u_{n+1}, Tu_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ &\quad +\rho \int_\Omega f^0(u; \beta(v - u_{n+1}))d\Omega \geq 0, \quad \forall v \in K_\beta, \end{aligned}$$

which is known as the proximal method for solving trifunction bihemivariational inequality (2.1).

In a similar way for $F(u, Tu, v-u) = \langle Au, v-u\rangle$, one can obtain a number of new and known proximal methods from Algorithm 3.11 for solving bihemivariational inequalities (2.2) and its special cases. This shows that the new methods suggested in this paper are unifying one and more general than the previous ones.

For the convergence analysis of Algorithm 3.11, we need the following result.

Theorem 3.5. *Let $\bar{u} \in K_\beta$ be a solution of (2.1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.10. If the trifunction $F(.,.,.)$ is pseudomonotone with respect to $\int_\Omega f^0(.;-))d\Omega$ and the operator $\int_\Omega f^0(.;.))d\Omega$ is monotone, then*

$$\begin{aligned} \|u_{n+1}-\bar{u}\|^2 \quad \leq \quad & \|u_n-\bar{u}\|^2 - \|u_{n+1}-u_n-\alpha_n(u_n-u_{n-1})\|^2 \\ & +\alpha_n\{\|u_n-\bar{u}\|^2 - \|u_{n-1}-\bar{u}\|^2 + 2\|u_n-u_{n-1}\|^2\}. \end{aligned} \tag{3.32}$$

Proof. Let $\bar{u} \in K_\beta$ be a solution of (2.1). Then

$$-F(v, Tv, \beta(\bar{u}-v)) + \int_\Omega f^0(\bar{u}; \beta(v-\bar{u})d\Omega \geq 0, \quad \forall v \in K_\beta, \tag{3.33}$$

since $F(.,.,.)$ is pseudomonotone with respect to $\int_\Omega f^0(.;.))d\Omega$.

Taking $v = u_{n+1}$ in (3.33), we have

$$F(u_{n+1}, Tu_{n+1}, \beta(\bar{u}-u_{n+1})) + \int_\Omega f^0(\bar{u}; \beta(\bar{u}-u_{n+1}))d\Omega \geq 0. \tag{3.34}$$

Now taking $v = \bar{u}$ in (3.31), we obtain

$$\begin{aligned} \rho F(u_{n+1}, Tu_{n+1}, \bar{u}-u_{n+1}) \quad & + \quad \langle u_{n+1}-u_n-\alpha_n(u_n-u_{n-1}), \bar{u}-u_{n+1}\rangle \\ & + \quad \rho\int_\Omega f^0(u_{n+1}; \beta(\bar{u}-u_{n+1}))d\Omega \geq 0. \end{aligned} \tag{3.35}$$

From (23), (24) and using the monotonicity of $\int_\Omega f^0(.;.))d\Omega$ we have

$$\begin{aligned}
& \langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), \bar{u} - u_{n+1}\rangle \\
\geq\ & -\rho F(u_{n+1}, Tu_{n+1}, \bar{u} - u_{n+1}) - \rho J^0(u_{n+1}; \hat{u} - u_{n+1}) \\
\geq\ & -\rho \int_\Omega f^0(u_{n+1}; \beta(\hat{u} - u_{n+1}))d\Omega \\
& + \int_\Omega f^0(\hat{u}; \beta(u_{n+1} - \hat{u}))d\Omega\} \geq 0,
\end{aligned} \tag{3.36}$$

which implies that

$$\langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle \geq \alpha_n \langle u_n - u_{n-1}, \bar{u} - u_n + u_n - u_{n+1}\rangle. \tag{3.37}$$

Using (3.25) and rearranging the terms in (3.37), one can easily obtain (3.32), the required result. □

Theorem 3.6. *Let H be a finite dimensional space. Let u_{n+1} be the approximate solution obtained from Algorithm 3.9 and $\bar{u} \in K_\beta$ be a solution of (2.1). If there exists $\alpha \in (0,1)$ such that $0 \leq \alpha_n \leq \alpha, \quad \forall n \in N$ and $\quad \sum_{n=1}^{\infty} \alpha_n \|u_n - u_{n-1}\|^2 \leq \infty$, then $\lim_{n\longrightarrow\infty} u_n = \bar{u}$.*

Proof. Let $\hat{u} \in K_\beta$ be a solution of (2.1). First we consider the case $\alpha_n = 0$. Using the technique of Theorem 3.3, we can prove that $\lim_{n\longrightarrow\infty} u_n = \hat{u}$.

Now we consider the case $\alpha_n > 0$. From (3.32), we have

$$\begin{aligned}
& \sum_{n+1}^{\infty} \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 \\
\leq\ & \|u_0 - \bar{u}\|^2 + \sum_{n=1}^{\infty} \{\alpha\|u_n - \bar{u}\|^2 + 2\|u_n - u_{n-1}\|^2\} \leq \infty,
\end{aligned}$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 = 0.$$

Repeating the arguments as in Theorem 3.3, one can easily show that $\lim_{n\to\infty} u_n = \hat{u}$, the required result. □

Conclusion

In this paper, we have introduced and studied the trifunction bihemivariational inequalities. Several special cases are discussed as applications of the trifunction bihemivariational inequalities. The auxiliary principle technique is used to suggest several implicit and explicit iterative methods for solving the trifunction bihemivariational inequalities, Convergence criteria of the proposed methods is discussed under suitable mild conditions. Results obtained in this paper continue to hold for the special cases. Comparison of the proposed methods with other methods need further efforts. The ideas and techniques of this paper stimulate further research in these dynamic fields

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Detecting Electronic Banking Fraud on Highly Imbalanced Data using Hidden Markov Models

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Abstract

Recent researches have revealed the capability of Machine Learning (ML) techniques to effectively detect fraud in electronic banking transactions since they have the potential to detect new and unknown intrusions. A major challenge in the application of ML to fraud detection is the presence of highly imbalanced data sets. In many available datasets, majority of transactions are genuine with an extremely small percentage of fraudulent ones. Designing an accurate and efficient fraud detection system that is low on false positives but detects fraudulent activity effectively is a significant challenge for researchers. In this paper, a framework based on Hidden Markov Models (HMM), modified Density Based Spatial Clustering of Applications with Noise (DBSCAN) and Synthetic Minority Oversampling Technique Techniques (SMOTE) is proposed to effectively detect fraud in a highly imbalanced electronic banking dataset. The various transaction types, transaction amounts and the frequency of transactions are taken into consideration by the proposed model to enable effective detection. With different number of hidden states for the proposed HMMs, simulations are performed for four (4) different approaches and their performances compared using precision, recall rate and F1-Score as the evaluation metrics. The study revealed that, our proposed approach is able to detect fraudulent transactions more effectively with reasonably low number of false positives.

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1. Introduction

E-banking is a form of banking where funds are transferred as exchange of electronic signals rather than cash, checks, or other types of paper documents [1]. Over the last few decades, E-Banking has redefined the way banking is conducted across the globe and the use of electronic payments platforms has continued to experience significant growth. It allows customers a 24-hour access to their accounts with the ability to transfer funds, perform on-line payments and apply for loans and other financial products virtually [2].

Fraud can be defined as any premeditated act of criminal deceit, trickery or falsification by a person or group of persons with the intention of altering facts, in order to obtain undue personal monetary advantage [3]. Unfortunately, fraud cases relating to cyber-crime perpetrated through E-banking resulted in an actual loss of GH¢14.31 million and therefore presents a unique challenge to individuals and financial institutions that offer those services [4]. To address this problem, financial institutions employ various fraud prevention tools such as real-time transaction authorization, transaction verification codes, transaction alerts, rule-based detection among others. Fraudsters however are adaptive, and given time, they devise several ways to circumvent such protection mechanisms [5]. There is therefore the need to implement enhanced technologies and systems that can detect fraud in real-time effectively in order to maintain the viability of these electronic payment systems where fraudsters constitute a very inventive and fast-moving fraternity. As preventive technology changes, so does the technology of criminals and the way they go about with their fraudulent activities [6]. While it is necessary to detect and possibly prevent fraudulent transactions, it is also very critical to ensure genuine transactions are executed successfully.

One of the most important techniques for intrusion/anomaly detection based on machine learning is using Hidden Markov Models (HMM) which are machine learning algorithms consisting of hidden states and observable outputs for modelling probability distributions over sequences of observations. The hidden state layer is a stable Markov chain and its state probability and state transition probability are decided from the initial state probability vector π and the state transition probabilities. Observable output layer is decided from the observed symbols probability matrix which is derived from the observed symbols of each hidden state [7].

The application of HMMs ranges from speech and image recognition, intrusion/fraud detection to motion/action analysis in videos among others and is generally characterized by the following [8];

1. The number of hidden states in the model denoted as *N*. The state at a specific time *t* is denoted by q_t.

2. The number of unique observation symbols denoted as *M*.

3. A transition probability between states denoted by a matrix $A = [a_{ij}]$, where:

$$a_{ij} = P(q_{t+1} = S_j \mid q_t = S_i). \tag{1}$$

Also,
$$\sum_{j=1}^{N} a_{ij} = 1, \quad 1 \leq i \leq N. \tag{2}$$

4. An emission probability matrix, $B = [b_j(k)]$, where

$$B_j(k) = P(V_k = q_t \mid S_j = q_t) \tag{3}$$

$$\sum_{k=1}^{M} b_j(k) = 1, \ \ 1 \leq j \leq N. \tag{4}$$

5. An initial probability for each state denoted by the vector $\pi = [\pi_i]$, where

$$\pi_i = P(q_1 = S_i), \quad \sum_{i=1}^{N} \pi_i = 1. \tag{5}$$

In recent decades, many research communities have been working toward HMM-based intrusion detection mainly because of its ability to detect new and unknown intrusions and usage in real-time applications by processing data streams on-the-fly. HMMs also allow for the usage of heterogeneous data sources as input, and visual representation of acquired knowledge relative to the other techniques of machine learning.

Over the past few years, the use of Electronic banking platforms has continued to experience significant growth and has redefined the way banking or E-commerce is conducted across the world [9]. On the other hand, fraudulent Electronic banking and E-commerce activities are becoming more and more sophisticated and challenging leading to massive financial losses. Effective and efficient detection of Electronic banking fraud is therefore regarded as one of the major challenges to all financial institutions, and is an increasing cause for concern [2].

According to the Bank of Ghana 2019 banking industry fraud report, fraud cases relating to cyber-crime perpetrated through electronic banking and mobile banking platforms accounts for the highest value of attempted fraud amounting to GH¢ 50.54

million with actual loss of GH¢14.31 million [4]. From available literature, majority of the works in the area of HMM-based fraud detection in Electronic banking focuses only on payments to merchants for goods and services . Transaction amounts are mostly taken as observation symbols and the types of items purchased considered as the hidden states of the proposed Hidden Markov Models. In related studies conducted by [10], [11], [12], [13], [14], and [15], techniques such as Neural Network , Bayesian Network , Dempster-Shafer theory, Support Vector Machine etc. are employed which incorporated other forms of electronic banking options such as remote funds transfers and deposits. However, all these proposed techniques perform classification based on a single transaction while relying on domain-expert features without considering a sequence of transactions to make a decision hence producing high levels of false positives.

A large number of false positives may translate into bad customer experience and may lead customers to take their business elsewhere. A major challenge in applying ML to fraud detection is presence of highly imbalanced data sets. In many available datasets, majority of transactions are genuine with an extremely small percentage of fraudulent ones. Designing an accurate and efficient fraud detection system that is low on false positives but detects fraudulent activity effectively is a significant challenge for researchers.

This proposed research seeks to develop and implement an improved fraud/intrusion detection system for both debit and credit transactions in electronic Banking using Hidden Markov Models by incorporating the various electronic banking platforms employed by customers, transaction amounts and the frequency at which these transactions occur. To determine the transaction profile of customers, the Density-based Spatial Clustering of Applications with Noise (DBSCAN) which is capable of discovering clusters of different shapes and sizes from a large amount of data containing noise and outliers was employed. Synthetic Minority Oversampling Technique (SMOTE) was also employed to handle the imbalanced class problem typical of Electronic banking datasets.

The rest of the paper is organized as follows: In Section 2, we present a review of related works. The methodology adopted for the study is outlined in Section 3. Detailed experimental results and discussion to establish the efficiency of the proposed approach is presented in Section 4. Finally, we conclude the paper with some discussions in Section 5.

2. Literature Review

Fraud Detection in Electronic Banking is understudied in literature perhaps due to security and data privacy concerns. We will begin by considering related works in electronic banking in general and then consider those specifically related to the use of credit cards which has been given considerable attention by researchers.

[10] presents a fraud detection system for online banking where differential analysis is used to obtain local evidence of fraud where a significant deviation from normal behavior indicates a potential fraud. The Dempster's rule of combination is applied to these evidences for final suspicion score of fraud. Their main contribution is a fraud detection method based on effective identification of devices used to access accounts and assessing the likelihood of being a fraud by tracking the number of different accounts accessed by each device. However, their system performs poorly for higher number of Hidden states and also when users' transaction patterns changes frequently.

[16] considered transaction amounts and purchases types as the emission symbols and hidden states respectively of the proposed HMM for online banking FDS. The model is trained with the normal behavior of an account holder using Baum-Welch algorithm and a One-time-Password is sent to the Customers contact number for authorization if an incoming transaction violates the behavior sequence. Although, the accuracy of their system was close to 72 percent over a wide variation in the input data, False Positives was still high especially when the transaction data is highly skewed. A fraudulent transaction could still go through if a fraudster has access to a customer's phone.

[11] incorporates several advanced data mining techniques for online banking fraud detection by building a contrast vector for each transaction based on its customer's historical behavior sequence. A novel algorithm, Contrast Miner, was introduced to efficiently mine contrast patterns and distinguish fraudulent from genuine behavior, followed by an effective pattern selection and risk scoring that combines predictions from different models. Results from experiments on large-scale real online banking data demonstrated that the proposed system achieves substantially higher accuracy and with lower false positives by incorporating domain knowledge and traditional fraud detection methods.

[12] rather modeled the sequence of operations in online banking transaction processing using HMMs and described how it could be used for the detection of frauds.

The observation sequence length is fixed to two (2) whilst changing sequence length

for training i.e., changing dataset length from 10 to 80 with difference of ten. Simulation results revealed that, although the complexity of the system also increases for increased observation sequence length, the accuracy of the proposed system is close to 60% with reduced false Positive rate.

The work done by [14] employed HMMs and k-means algorithm for detecting fraud in online banking transactions. In their proposed model, a variable is used to keep the number of transactions within a period of time before and after each transaction as well as the quantified amounts as the observation symbols. If an incoming transaction is not accepted by the trained HMM with sufficiently high probability, it is considered fraudulent. The feasibility of their proposed model is demonstrated through simulation experiments using real-world bank transaction data. In the case of enough historical transactions, their model performs well for low, medium frequency and amount of user groups. An efficient Prior determination of the number of clusters is considered a major challenge in their proposed approach.

Specifically on fraud detection relating to the use of Credit Cards, [17] considered purchase types and transaction amounts as hidden states and observation symbols respectively in their proposed HMM. In order to estimate the model parameters, the K-means clustering algorithm is employed to determine the spending profile of cardholders. An incoming transaction is considered fraud if it is not accepted by the HMM with a significantly high probability. Experimental results revealed that, their proposed model recorded an accuracy close to eighty (80) percent over a wide variation of the data. An efficient prior determination of the number of clusters and significant number of false positives were considered the major challenges in their proposed approach.

[18] performed a comparative analysis of intrusion detection models on highly skewed credit card data based on Decision Trees, Random Forest, Support Vector Machines (SVM) and logistic regression. The original sample was randomly partitioned into k-equal sized subsamples where a single subsample is retained as the validation data for testing the model, and the remaining $k-1$ subsamples used as training data. With the four basic metrics employed, namely True positive (TPR), True Negative (TNR), False Positive (FPR) and False Negative (FNR) rates, Simulation results using dataset provided by ULB machine learning revealed that, Logistic regression and Random forest shows the most precise and high accuracy in the area of credit card fraud detection but requires very large dataset for training and also suffers from the imbalanced dataset problem even after preprocessing.

In order to reduce the number of false positives, [19] proposed a model based on automated feature engineering to automatically derive behavioral features based on the historical data of a credit card associated with a transaction. A total of 237 features for each transaction was generated, and a random forest was then employed to learn a classifier. One important feature of their proposed model is that, it also utilizes the distance between two locations transactions on an account has occurred and whether they occurred in person or remotely is established. The proposed model was tested on data from a large multinational bank and compared to existing solutions and revealed that, on an unseen data of 1.852 million transactions, false positives was reduced by about 54%. However, since their models Perform classification based on a single transaction there was a performance degradation when transaction pattern of users changes frequently.

3. Methodology

There is generally a very limited number of public datasets on electronic banking for research purposes mainly due to personal and security concerns. In this research, a Kaggle provided dataset of simulated mobile based transactions is adopted. As detailed in Table 1, the dataset is highly imbalanced due to the fact that only 8,312 transactions out of the almost 6 million transactions are labeled as fraud.

Table 1: Details of the Paysim Dataset adopted for the study.

Transaction Type	**# of Genuine Transactions**	**# of Fraudulent Transactions**	**Total**
TRANSFER	528812	4097	532909
CASH-OUT	2233384	4116	2237500
CASH-IN	1399284	0	1399284
DEBIT	41432	0	41432
PAYMENTS	2151494	0	2151494
TOTAL	6354407	8213	6362620

'CASH IN' and 'CASH OUT' represents an increase in account balance of a customer as a result of cash inflow and a decrease in account balance as a result of cash outflow respectively. 'TRANSFER' refers to movement of money between users whilst 'PAYMENT' represents the settlements made for goods and services to merchants. 'DEBIT' as used in this context signifies the sending of money from a mobile service (electronic wallet) to a bank account.

3.1. Data pre-processing

To effectively evaluate the performance of our proposed models on the highly class imbalanced dataset, Synthetic Minority Oversampling Technique (SMOTE) is employed to generate virtual training records by linear interpolation for the fraudulent transactions by randomly selecting one or more of the k-nearest neighbors for each specific fraudulent transaction. After the oversampling process, the data is reconstructed and then the proposed Hidden Markov Models is applied on the processed data. Specifically, the sampling rate is set to 73000 %.

The proposed SMOTE technique as adopted in this study is presented in Algorithm 1.

Algorithm 1: The *SMOTE* algorithm

Procedure SMOTE (f, R, k)

Input: Number of Fraudulent Transactions (f); Amount of SMOTE R %; Number of nearest neighbors k

Output: $(R/100) * f$ synthetic Fraudulent Transactions

1: The number of Fraudulent Transactions is set to f

2: For each $y \in f$, the k-nearest neighbors of y are obtained by calculating the Euclidean distance between y and every other sample in set f.

3: The sampling rate R is set according to the imbalanced proportion.

4: For each $y \in f$, R examples are randomly selected from its k-nearest neighbors, and they construct the set f_1.

5: For each example $y_t \in f_1$ $(t = 1, 2, 3, \ldots, R)$, generate a new sample as in equation 6 below:

$$y' = y + rand(0, 1) * |y - y_t|. \tag{6}$$

3.2. Identifying transaction profile of customers

For optimal training of the our proposed Hidden Markov Models, a modified Density Based Spatial Clustering of Applications with Noise (DBSCAN) and the K-means clustering algorithms are executed on each customer's previous transactions by considering the amount and frequency of transactions. K-means is an unsupervised learning algorithm for grouping a given set of data based on their similarity where the numbers of clusters are fixed a priori. The grouping is performed by minimizing the sum

of squares of distances between each data point and the centroid of the cluster to which it belongs to [20]. The DBSCAN clustering technique however filters out outliers and discovers clusters of arbitrary shapes [21]. We modified the DBSCAN algorithm by adding a step that computes the centroid of each cluster later to be used to dynamically convert an incoming transaction into an observation symbol in the fraud detection process.

The proposed DBSCAN technique as adopted in this study is presented as in Algorithm 2.

Algorithm 2: The DBSCAN algorithm.

DBSCAN (dataset, *d*, minpts)

Input: A set of points, *dataset*, distance threshold *d*, and the minimum number of points required to form a cluster, min*pts*.

Output: A set of clusters representing the various observation symbols

1: $n = 1$, #initialise the cluster index to 1

2: For each unvisited point *pt* in dataset, mark *pt* as visited

3: Find the neighboring points, *N* of *p*

4: If $|N|$ >=minpts then $N = N \: U \: N'$

5: if p' is not a member of any cluster, Mark it as noise.

6: Compute the centroid $\hbar$ of each cluster using (7) below

$$\hbar = \frac{1}{n_i} \sum_{x_j \in c_i} x_j, \tag{7}$$

where n_i is the number of points in cluster c_i.

Spending profiles of accountholders are determined at the end of the clustering step. Let ψ_i be the percentage of total number of transactions of an accountholder, then, the spending profile ρ of an account holder, ϑ is determined as in (8):

$$\rho(\vartheta) = \arg \underset{i}{Max}(\psi_i). \tag{8}$$

The cluster number to which most of the transactions of the account holder belongs to represents the spending profile of the account holder. The computed centroids are used

to generate the observation symbol for a new transaction $\emptyset$ (denoted by $\emptyset_m$) is defined as in (9).

$$\phi_m = v_{arg_i} min|m - n_i|. \tag{9}$$

The i[th] transaction on account A_k denoted as $P_{i,y}^{A_k}$ is suspected to be an outlier if it does not belong to any cluster in the set C' where *y* refers to the frequency of transaction. If the average distance of the amount p of an outlier transaction $P_{i,y}^{A_k}$ from the set of existing clusters in C' is W_a, then its level of deviation o_l is given as in (10):

$$o_l = \begin{cases} \frac{W_a - \epsilon}{W_a} & if\ |N_\epsilon(p)| < MinPts \\ 0 & otherwise. \end{cases} \tag{10}$$

The key idea of the modified DBSCAN algorithm is that for each point *p* in a cluster C_i, there are at least a minimum number of points (*MinPts*) in the e-neighborhood of that point p denoted as $N_\varepsilon(p)$ i.e., the density in the e-neighborhood has to exceed some threshold. The proposed K-means algorithm as adopted in this study is presented in Algorithm 3.

Algorithm 3: K-Means Clustering Algorithm

1: Specify the number of clusters to assign

2: Randomly initialize *k* centroids

3: **Repeat**

4: **Expectation**: Assign each point to its closest centroid

5: **Maximization**: Compute the new Centroid (Mean) of each cluster

6: Until the Centroid Positions do not change

Specifically, for this research, the set C = {low-frequency low-amount, low-frequency medium-amount, low-frequency high-amount, medium frequency low-amount, medium-frequency medium-amount, medium-frequency high-amount, high-frequency low- amount, high-frequency medium-amount, high-frequency high-amount} denotes the clusters.

The set R = {transaction_amount, frequency_of_transaction} represents the set of attributes used to generate these clusters.

To compute the probability of an observed sequence, $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{T-1})$ with

respect to our Hidden Markov Model λ, where $\upsilon = (\upsilon_0, \upsilon_1, \upsilon_2, \ldots, \upsilon_{T-1})$ represents the various hidden states, the definition of the emission transition Matrix is defined as in (11);

$$P(\sigma|\upsilon, \lambda) = b_{\upsilon 0}(\sigma_o) b_{\upsilon 1}(\sigma_1) \ldots b_{\upsilon T-1}(\sigma_{T-1}). \tag{11}$$

The Initial transition vector, π and State Transition Matrix, A are also defined as in (12) and (13);

$$P(\upsilon|\lambda) = \pi_\upsilon a_{\upsilon,\upsilon 1} \ldots a_{\upsilon T-2,\upsilon T-1} \tag{12}$$

and

$$P(\sigma, \upsilon|\lambda) = P(\sigma|\upsilon, \lambda) P(\upsilon|\lambda). \tag{13}$$

By summing over all possible state sequences, (14) through (16) is obtained

$$P(\sigma|\upsilon) = \sum P(\sigma, \upsilon|\lambda)_X \tag{14}$$

$$= \sum P(\sigma|\upsilon, \lambda) P(\upsilon|\lambda)_\upsilon \tag{15}$$

$$= \sum \pi_{\upsilon 0} b_{\upsilon 0}(\sigma_o) a_{\upsilon,\upsilon 1} \ldots a_{\upsilon T-2,\upsilon T-1} b_{\upsilon T-1}(\sigma_T - 1)_\upsilon. \tag{16}$$

The probability of the observation sequences denoted as $e_t(i)$, where the system is in state q_i at time *t* is defined in (17).

$$e_t(i) = P(\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_t, \upsilon_t = q_i|\lambda). \tag{17}$$

$e_t(i)$ is then calculated recursively as in (18) to (20):

1. Let $e_0(i) = \pi_i b_i(\sigma_0)$ (18)

2. For $i = 0, 1, 2, \ldots, N-1$ and $t = 1, 2, \ldots, T-1$, we compute

$$e_t(i) = \sum_{j=0}^{N-1} [e_{t-1}(j) a_{ji}]\, b_i(\sigma_t) \tag{19}$$

3. Equation (20) is obtained from (19)

$$P(\sigma|\lambda) = \sum_{i=0}^{N-1} e_{T-1}(i). \tag{20}$$

3.3. Training the proposed HMMs

The transaction amounts are categorized into a Low $(l) = (0;\ 100]$, Medium $(m) = (100;\ 500]$, and High $(h) = (500;$ Transaction Limit$]$ values. The frequency at which these transactions occur on a particular is also categorized into a Low (Less than 5 times a month), Intermediate (Between 5 and 10 times a month), and High (at least 10 times a month) are also considered by our proposed model. For example, if an

accountholder performs about seven (7) transactions with the month with an average value of say 300, then the corresponding observation symbol is medium-frequency medium-amount (mm).

The various transaction types are considered the internal states whilst the transaction amounts combined with the frequency at which they occur denoted as $\{ll, lm, lh, ml, mm, mh, hl, hm, hh\}$ represents the observation symbols of our proposed Hidden Markov Model.

After formulating the hidden states and observation symbol, a hybrid optimization algorithm as presented in Algorithm 4 comprising the Baum-Welch, Particle Swam and Genetic Algorithms is used to effectively train the proposed models.

Algorithm 4: A hybrid algorithm for optimizing the parameters of the proposed HMMs

1. Initialize the parameters (A, B, π) using the spending profile of the customer
2. $\alpha_t(i) = \sum_{j=0}^{N-1}[\alpha_{t-1}(j)a_{ji}]\, b_i(O_t)$ (21)
3. $\beta_t(i) = \sum_{j=0}^{N-1}[\beta_{t+1}(j)a_{ij}]\, b_j(O_{t+1})$ (22)
4. $\gamma_t(i) = \frac{\alpha_t(i)\beta_t(i)}{P(O|\lambda)}$ (23)
5. $\gamma_t(i, j) = \frac{\alpha_t(i)a_{ij}b_j(O_{t+1})\beta_{t+1}(j)}{P(O|\lambda)}$ (24)
6. For $0 \leq i, j \leq N - 1$
7. $\pi_i = \gamma_0(i)$ (25)
8. $a_{ij} = \sum_{t=0}^{T-2} \gamma_t\,(i, j) / \sum_{t=0}^{T-2} \gamma_t\,(i)$ (26)
9. $b_j(k) = \sum_{\substack{t\epsilon\{0,1,\ldots T-1\} \\ O_t=k}} \gamma_t\,(j) / \sum_{t=0}^{T-1} \gamma_t\,(i)$ (27)
10. Go to 6
11. After 100 iterations, each solution becomes a chromosome for the genetic procedure and the fitness function $P(O|\lambda)$ is applied.
12. A multiple point crossover and mutation is performed to select the best 50 solutions for the next generation which are then positioned as particles in a search space using the PSO technique.
13. The position and velocity of each particle during each iteration is updated using;
14. $X_i(t + 1) = X_i(t) + V_i(t)$ (28)
15. $V_i(t + 1) = wV_i(t) + r_1\, \mathrm{U}([0,1])\, (X + i\,(t) - X_i(t))$

$$+ r_2\, \mathrm{U}([0,1]) \left(\hat{x}_i(t) - X_i(t) \right). \quad (29)$$

16. Compare the best solution each particle and the best position of the entire group and make appropriate adjustments

17. Termination Criteria Reached? If yes go to 18, otherwise go to 13

18. Output A, B and π

3.4. Fraud detection

To effectively classify an incoming transaction as fraudulent or otherwise, sequence of observation symbols, say $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_r$ are extracted from the training data of an account holder and its probability of acceptance, ∂_1 is computed by the model as in (30) by employing (20)

$$\partial_1 = p(\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_r | \lambda). \tag{30}$$

An incoming transaction occurring at time is converted to an observation symbol denoted as σ_{t+1} using (9) is used to replace the first observation symbol, σ_1 and its probability of acceptance by the model denoted as ∂_2 is also computed as in (31).

$$\partial_2 = p(\sigma_2, \sigma_3, \sigma_4, \ldots \sigma_{r+1} | \lambda). \tag{31}$$

The newly generated transaction is classified as fraud and if the difference between ∂_1 and ∂_1 denoted as $\Delta\partial$ is above a predefined threshold (ε) as in (32).

$$\Delta\partial/\partial_1 \geq \varepsilon. \tag{32}$$

A genuine transaction is added to the sequence permanently to contribute to determining the validity or otherwise of the next transaction since transaction behavior of an accountholder could be dynamic. Otherwise, the transaction is declined, and the symbol is discarded.

Due to the highly imbalanced nature of the dataset, precision(p), recall(R) and F1-scores (F) as presented in equations (33) to (35) respectively are used as evaluation metrics (Wedge et al. [19]). Tp, Fp, Fn represent True Positives, False Positives and False Negatives respectively.

$$p = Tp/(Tp + Fp) \tag{33}$$

$$R = Tp/(Tp + Fn) \tag{34}$$

$$F = 2 * (P * R)/(P + R). \tag{35}$$

Precision quantifies the number of correct positive predictions made whilst Recall refers to the number of correct positive predictions made out of all possible positive

predictions. F-Measure however provides a way to combine both precision and recall into a single measure.

4. Simulation Results and Discussion

For different number of hidden states, four (4) sets of simulations were performed in two (2) stages using Python programming and their performance compared. For all the four sets of experiments, the proposed hidden Markov models were executed in the second stage. In the first stages of the first and second set of experiments, K-means and the modified DBSCAN algorithms were executed respectively. In first stage of the third set of experiment, both SMOTE and K-means techniques were employed whereas SMOTE and the modified DBSCAN clustering techniques were executed during the last set of experiments. The dataset was loaded and divided into two, 80% of it is used for training and evaluation whilst the rest is held back for validation.

4.1. Precision comparison

The precision of the four approaches is presented for different number of hidden states and presented in Figure 1. It is very clear that our proposed approach (SMOTE+DBSCAN+HMM) performed better for the various hidden states. Applying only the modified DBSCAN clustering technique with Hidden Markov Models performed relatively better than that of employing K-Means. It is also worth noting that, relatively higher values of precision scores were recorded when the SMOTE technique is adopted.

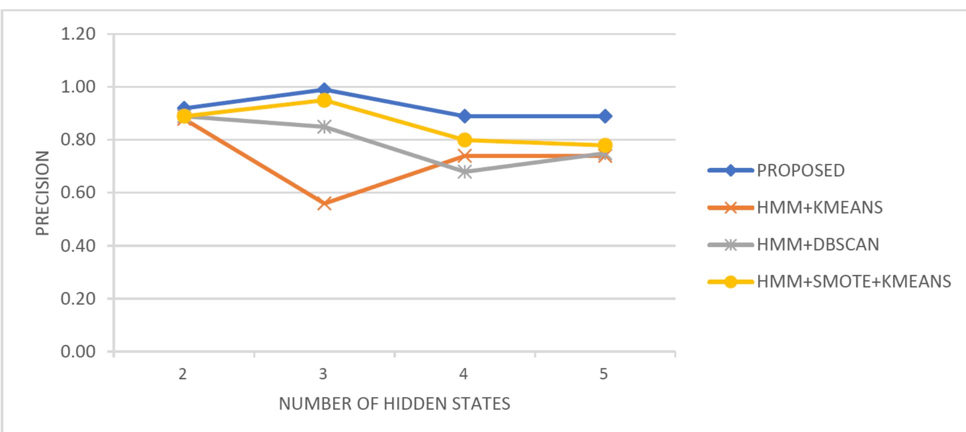

Figure 1: Comparison of the precision of the four (4) different approaches for different number of hidden states.

4.2. Recall rates comparison

A comparison of the Recall rates of the four (4) different approaches for different numbers of hidden states are presented in Figure 2. It is evident that approaches that employed the SMOTE technique appear to perform relatively better. Similarly the modified DBSCAN clustering technique performed better as compared to the K-means.

It can also be observed that, for higher values of N, all the approaches performed well except for those that employed Only K-means and Hidden Markov models without handling the class imbalance classification.

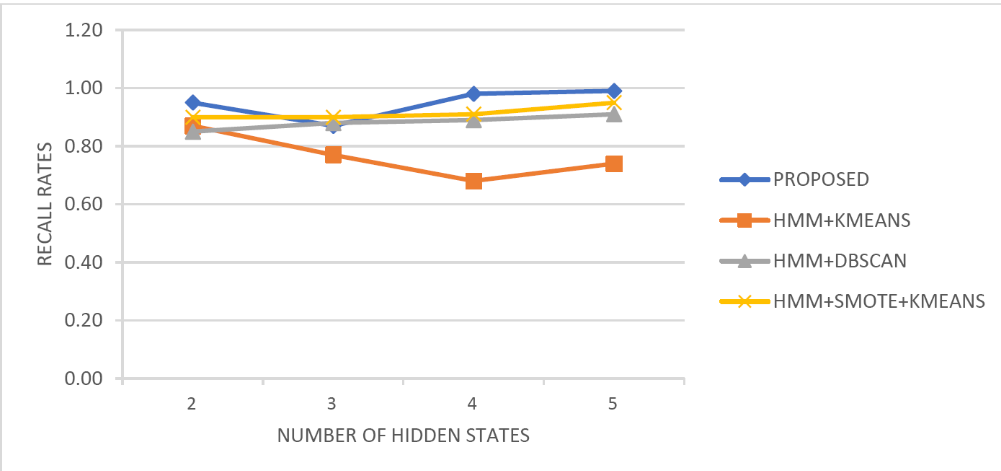

Figure 2: Recall rates of the four (4) approaches for different number of Hidden states.

4.3. F-measure comparison

In Figure 3, the F1-score for the four approaches are presented various Hidden states. It is observed that, higher F1-scores are obtained when the modified DBSCAN clustering technique is used as compared to using the K-means. Also, approaches that incorporated the SMOTE technique performed better. Employing both SMOTE and the modified DBSCAN clustering algorithms appears to perform relatively better than the other. All approaches performed relatively better when the number of hidden states is 3.

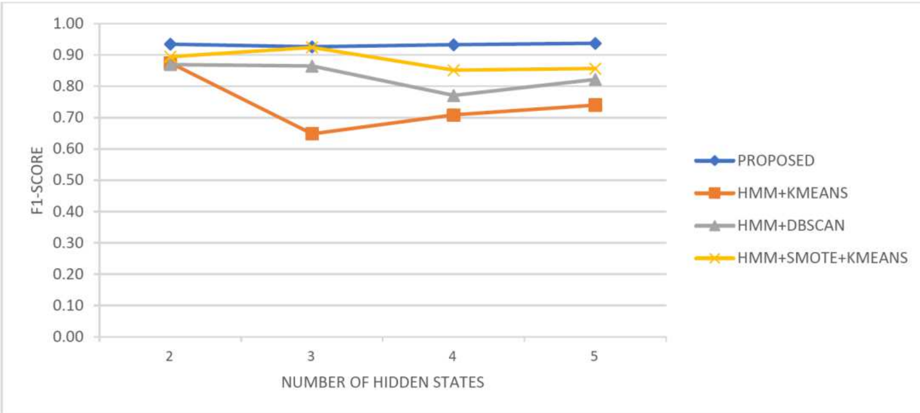

Figure 3: F1-scores of the four (4) different approaches for various number of Hidden states.

5. Conclusion

In this research, an improved electronic banking fraud detection framework based on Hidden Markov Models (HMM) and modified Density Based Spatial Clustering of Applications with Noise (DBSCAN) is proposed and implemented. The Synthetic Minority Oversampling Technique (SMOTE) is also employed due to the highly class imbalance nature of the dataset adopted. With different numbers of hidden states, simulations were performed two stages for four (4) different approaches in Python and their performance compared. For all the four sets of experiments, the proposed hidden Markov models were executed in the second stage. In the first stages of the first and second set of experiments, K-means and the modified DBSCAN algorithms were executed respectively. In first stage of the third set of experiment, both SMOTE and K-means techniques were employed whereas SMOTE and the modified DBSCAN clustering techniques were executed during the last set of experiments.

Generally, our proposed approach (SMOTE+DBSCAN+HMM) performed relatively better for all the various hidden states in terms of precision, recall and F1-Scores. Employing the modified DBSCAN clustering technique to determine the spending profile of customers and subsequently performed relatively better than using the K-Means algorithm since it filters out most of the easily recognizable fraudulent transactions before the proposed HMMs are applied. It is also evident from the simulation analysis that, the SMOTE technique effectively handles the class imbalance classification necessary to achieve improved performance.

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Generalized Oresme Numbers

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Abstract

In this paper, we introduce the generalized Oresme sequence and we deal with, in detail, three special cases which we call them modified Oresme, Oresme-Lucas and Oresme sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

1 Introduction

The Oresme sequence, $\{O_n\}_{n\geq 0}$, was introduced by Nicole Oresme (1320–1382) in the 14-th century. Oresme found the sum of the rational numbers formed by the terms $0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, ..., \frac{n}{2^n}$. These numbers form a second order sequence and are defined by the recurence relation

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, \qquad O_0 = 0, O_1 = \frac{1}{2}.$$

In [4], Horadam presented a history and obtained an abundance of properties of these numbers. Oresme numbers have many interesting properties and applications in many fields of science (see, for example, [1,2,3,4,9]).

The purpose of this article is to generalize and investigate these interesting sequence of numbers (Oresme numbers). First, we recall some properties of Fibonacci numbers and its generalizations, namely generalized Fibonacci numbers.

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The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers $\{F_n\}_{n\geq 0}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \; F_1 = 1,$$

and the sequence of Lucas numbers $\{L_n\}_{n\geq 0}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \; L_1 = 1.$$

The generalization of Fibonacci sequence leads to several nice and interesting sequences. The generalized Fibonacci sequence (or generalized (r, s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0, W_1; r, s)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined (by Horadam [6]) as follows:

$$W_n = rW_{n-1} + sW_{n-2}, \qquad W_0 = a, W_1 = b, \quad n \geq 2 \tag{1.1}$$

where W_0, W_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [5,7,8] and Soykan [12].

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1: A few special case of generalized Fibonacci sequences.

Name of sequence	$W_n(a, b; r, s)$	Binet Formula	OEIS[10]
Fibonacci	$W_n(0, 1; 1, 1) = F_n$	$\dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$	A000045
Lucas	$W_n(2, 1; 1, 1) = L_n$	$\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$	A000032
Pell	$W_n(0, 1; 2, 1) = P_n$	$\dfrac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$	A000129
Pell-Lucas	$W_n(2, 2; 2, 1) = Q_n$	$(1+\sqrt{2})^n + (1-\sqrt{2})^n$	A002203
Jacobsthal	$W_n(0, 1; 1, 2) = J_n$	$\frac{2^n-(-1)^n}{3}$	A001045
Jacobsthal-Lucas	$W_n(2, 1; 1, 2) = j_n$	$2^n + (-1)^n$	A014551

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

Now we define two special cases of the sequence $\{W_n\}$. (r, s) sequence $\{G_n(0, 1; r, s)\}_{n\geq 0}$ and Lucas (r, s) sequence $\{H_n(2, r; r, s)\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = rG_{n+1} + sG_n, \qquad G_0 = 0, G_1 = 1, \tag{1.2}$$

$$H_{n+2} = rH_{n+1} + sH_n, \qquad H_0 = 2, H_1 = r, \tag{1.3}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{r}{s}G_{-(n-1)} + \frac{1}{s}G_{-(n-2)}, \\ H_{-n} &= -\frac{r}{s}H_{-(n-1)} + \frac{1}{s}H_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2)-(1.3) hold for all integer n.

Some special cases of (r, s) sequence $\{G_n(0, 1; r, s)\}_{n\geq 0}$ and Lucas (r, s) sequence $\{H_n(2, r; r, s)\}_{n\geq 0}$ are as follows:

1. $G_n(0, 1; 1, 1) = F_n$, Fibonacci sequence,
2. $H_n(2, 1; 1, 1) = L_n$, Lucas sequence,
3. $G_n(0, 1; 2, 1) = P_n$, Pell sequence,
4. $H_n(2, 2; 2, 1) = Q_n$, Pell-Lucas sequence,
5. $G_n(0, 1; 1, 2) = J_n$, Jacobsthal sequence,
6. $H_n(2, 1; 1, 2) = j_n$, Jacobsthal-Lucas sequence.

The following theorem shows that the generalized Fibonacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1. *For $n \in \mathbb{Z}$, for the generalized Fibonacci sequence (or generalized (r,s)-sequence or Horadam sequence or 2-step Fibonacci sequence), we have the following:*

(a)

$$\begin{aligned} W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0) \\ &= (-1)^{n+1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

(b)

$$W_{-n} = \frac{(-1)^{n+1}s^{-n}}{-W_1^2 + sW_0^2 + rW_0W_1}((2W_1 - rW_0)W_0W_{n+1} - (W_1^2 + sW_0^2)W_n).$$

Proof. For the proof, see Soykan [13, Theorem 3.2 and Theorem 3.3]. □

The following theorem presents sum formulas of generalized (r,s) numbers (generalized Fibonacci numbers).

Theorem 2. *Let x be a real (or complex) number. For all integers m and j, for generalized (r,s) numbers (generalized Fibonacci numbers), we have the following sum formulas:*

(a) *If $(-s)^m x^2 - xH_m + 1 \neq 0$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{\begin{array}{c}((-s)^m x - H_m)x^{n+1}W_{mn+j} \\ + (-s)^m x^{n+1}W_{mn+j-m} + W_j - (-s)^m xW_{j-m}\end{array}}{(-s)^m x^2 - xH_m + 1}. \quad (1.4)$$

(b) *If $(-s)^m x^2 - xH_m + 1 = u(x-a)(x-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{\begin{array}{c}(x(n+2)(-s)^m - (n+1)H_m)x^n W_{j+mn} \\ + (-s)^m (n+1)x^n W_{mn+j-m} - (-s)^m W_{j-m}\end{array}}{2(-s)^m x - H_m}.$$

(c) *If* $(-s)^m x^2 - xH_m + 1 = u(x-c)^2 = 0$ *for some* $u, c \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $x = c$, *then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{\begin{array}{r}(n+1)\left((-s)^m (n+2)x^n - nx^{n-1}H_m\right) W_{mn+j} \\ + n(n+1)(-s)^m x^{n-1} W_{mn+j-m}\end{array}}{2(-s)^m}.$$

Proof. It is given in Soykan [13, Theorem 4.1]. □

Note that (1.4) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{\begin{array}{r}((-s)^m x - H_m)x^{n+1} W_{mn+j} + (-s)^m x^{n+1} W_{mn+j-m} \\ + x(H_m - (-s)^m x)W_j - (-s)^m x W_{j-m}\end{array}}{(-s)^m x^2 - xH_m + 1}.$$

We give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 3. *Suppose that* $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized Fibonacci sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum\limits_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \tag{1.5}$$

Proof. For a proof, see [12, Lemma 1.1]. □

Binet's formula of generalized Fibonacci sequence can be calculated using its characteristic equation (the quadratic equation) which is given as

$$x^2 - rx - s = 0. \tag{1.6}$$

The roots of characteristic equation are

$$\alpha = \frac{r + \sqrt{\Delta}}{2}, \quad \beta = \frac{r - \sqrt{\Delta}}{2}. \tag{1.7}$$

where

$$\Delta = r^2 + 4s$$

and the followings hold

$$\begin{aligned}\alpha+\beta &= r, \\ \alpha\beta &= -s, \\ (\alpha-\beta)^2 &= (\alpha+\beta)^2-4\alpha\beta = r^2+4s.\end{aligned}$$

1.1 Binet's Formula for the Distinct Roots Case

In this subsection, we assume that the roots α and β of characteristic equation (1.6) are distinct. Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 4 (Distinct Roots Case)**.** *Binet's formula of generalized Fibonacci numbers is*

$$W_n = \frac{b_1\alpha^n}{\alpha-\beta} + \frac{b_2\beta^n}{\beta-\alpha} = \frac{b_1\alpha^n - b_2\beta^n}{\alpha-\beta} \tag{1.8}$$

where

$$b_1 = W_1 - \beta W_0,\ b_2 = W_1 - \alpha W_0.$$

(1.8) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n \tag{1.9}$$

where

$$A_1 = \frac{W_1-\beta W_0}{\alpha-\beta},\ A_2 = \frac{W_1-\alpha W_0}{\beta-\alpha}.$$

Note that

$$\begin{aligned}A_1A_2 &= \frac{(W_1^2 - sW_0^2 - rW_1W_0)}{-(r^2+4s)}, \\ A_1+A_2 &= W_0.\end{aligned}$$

We next find Binet's formula of generalized Fibonacci numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 5 (Binet's formula of generalized Fibonacci numbers)**.**

$$W_n = \frac{d_1\alpha^n}{(\alpha-\beta)} + \frac{d_2\beta^n}{(\beta-\alpha)} \tag{1.10}$$

where

$$\begin{aligned} d_1 &= W_0\alpha + (W_1 - rW_0), \\ d_2 &= W_0\beta + (W_1 - rW_0)\beta. \end{aligned}$$

Proof. For a proof, see [12, Theorem 1.2]. □

Note that from (1.8) and (1.10) we have

$$W_1 - \beta W_0 = W_0\alpha + (W_1 - rW_0), \tag{1.11}$$

$$W_1 - \alpha W_0 = W_0\beta + (W_1 - rW_0)\beta. \tag{1.12}$$

For all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in (1.8) or (1.10)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)}, \\ H_n &= \alpha^n + \beta^n, \end{aligned}$$

respectively.

1.2 Binet's Formula for the Single Root Case

In this subsection, we assume that the roots α and β of characteristic equation (1.6) are equal, i.e., $\alpha = \beta$. So (1.6) can be written as

$$x^2 - rx - s = (x-\alpha)^2 = x^2 - 2\alpha x + \alpha^2 = 0.$$

Note that in this case,

$$\begin{aligned} \alpha &= \frac{r}{2}, \\ r &= 2\alpha, \\ s &= -\alpha^2 = -\frac{r^2}{4}, \\ r^2 + 4s &= 0. \end{aligned}$$

Using the root α and the recurrence relation, Binet's formula can be given as follows:

Theorem 6 (Single Root Case)**.** *Binet's formula of generalized Fibonacci numbers is*

$$W_n = (D_1 + D_2 n)\alpha^n \tag{1.13}$$

where

$$\begin{aligned} D_1 &= W_0, \\ D_2 &= \frac{1}{\alpha}\left(W_1 - \alpha W_0\right). \end{aligned}$$

Proof. For a proof, see Soykan [13]. □

Note that (1.13) can be written as

$$\begin{aligned} W_n &= (W_0 + \frac{1}{\alpha}\left(W_1 - \alpha W_0\right) n)\alpha^n \\ &= \left(nW_1 - \alpha\left(n-1\right)W_0\right)\alpha^{n-1} \\ &= \left(nW_1 - \frac{r}{2}\left(n-1\right)W_0\right)\left(\frac{r}{2}\right)^{n-1}. \end{aligned}$$

We also see that

$$\begin{aligned} D_1 D_2 &= \frac{W_0(2W_1 - rW_0)}{r}, \\ D_1 + D_2 &= 2\frac{W_1}{r}. \end{aligned}$$

For all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in (1.8) or (1.10)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= n\alpha^{n-1}, \\ H_n &= 2\alpha^n, \end{aligned}$$

respectively.

2 Generalized Oresme Sequence

In this paper we consider the case $r = 1, s = -\frac{1}{4}$. A generalized Oresme sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 4W_{-(n-1)} - 4W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integers n.

Eq. (1.13) can be used to obtain Binet formula of generalized Oresme numbers. Binet formula of generalized Oresme numbers can be given as

$$W_n = (D_1 + D_2 n)\alpha^n \tag{2.2}$$

where

$$\begin{aligned} D_1 &= W_0, \\ D_2 &= \frac{1}{\alpha}(W_1 - \alpha W_0). \end{aligned}$$

i.e.,

$$W_n = (W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)n)\alpha^n.$$

Here, $\alpha = \beta = \frac{1}{2}$ are the roots of the quadratic equation

$$x^2 - x + \frac{1}{4} = 0. \tag{2.3}$$

i.e. the roots of characteristic equation (2.3) are equal. Note that

$$\begin{aligned} \alpha + \beta &= 1, \\ \alpha\beta &= \frac{1}{4}, \\ \alpha - \beta &= 0. \end{aligned}$$

and

$$W_n = (W_0 + 2\left(W_1 - \frac{1}{2}W_0\right)n) \times \frac{1}{2^n}.$$

The first few generalized Oresme numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2: A few generalized Oresme numbers.

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$4W_0 - 4W_1$
2	$W_1 - \frac{1}{4}W_0$	$12W_0 - 16W_1$
3	$\frac{3}{4}W_1 - \frac{1}{4}W_0$	$32W_0 - 48W_1$
4	$\frac{1}{2}W_1 - \frac{3}{16}W_0$	$80W_0 - 128W_1$
5	$\frac{5}{16}W_1 - \frac{1}{8}W_0$	$192W_0 - 320W_1$
6	$\frac{3}{16}W_1 - \frac{5}{64}W_0$	$448W_0 - 768W_1$
7	$\frac{7}{64}W_1 - \frac{3}{64}W_0$	$1024W_0 - 1792W_1$
8	$\frac{1}{16}W_1 - \frac{7}{256}W_0$	$2304W_0 - 4096W_1$
9	$\frac{9}{256}W_1 - \frac{1}{64}W_0$	$5120W_0 - 9216W_1$
10	$\frac{5}{256}W_1 - \frac{9}{1024}W_0$	$11264W_0 - 20480W_1$

Now we define three special cases of the sequence $\{W_n\}$. Modified Oresme sequence $\{G_n\}_{n\geq 0}$, Oresme-Lucas sequence $\{H_n\}_{n\geq 0}$ and Oresme sequence $\{O_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = G_{n+1} - \frac{1}{4}G_n, \qquad G_0 = 0, G_1 = 1, \tag{2.4}$$

$$H_{n+2} = H_{n+1} - \frac{1}{4}H_n, \qquad H_0 = 2, H_1 = 1, \tag{2.5}$$

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, \qquad O_0 = 0, O_1 = \frac{1}{2}. \tag{2.6}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{O_n\}_{n\geq 0}$ can be extended to negative

subscripts by defining

$$\begin{aligned} G_{-n} &= 4G_{-(n-1)} - 4G_{-(n-2)}, \\ H_{-n} &= 4H_{-(n-1)} - 4H_{-(n-2)}, \\ O_{-n} &= 4O_{-(n-1)} - 4O_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.4)-(2.6) hold for all integers n.

Next, we present the first few values of the modified Oresme, Oresme-Lucas and Oresme numbers with positive and negative subscripts:

Table 3: The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11
G_n	0	1	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{7}{64}$	$\frac{1}{16}$	$\frac{9}{256}$	$\frac{5}{256}$	$\frac{11}{1024}$
G_{-n}	....	-4	-16	-48	-128	-320	-768	-1792	-4096	-9216	-20480	-45056
H_n	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$
H_{-n}	....	4	8	16	32	64	128	256	512	1024	2048	4096
O_n	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{5}{32}$	$\frac{3}{32}$	$\frac{7}{128}$	$\frac{1}{32}$	$\frac{9}{512}$	$\frac{5}{512}$	$\frac{11}{2048}$
O_{-n}	....	-2	-8	-24	-64	-160	-384	-896	-2048	-4608	-10240	-22528

For all integers n, modified Oresme, Oresme-Lucas and Oresme numbers (using initial conditions in (2.2)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= n\alpha^{n-1} = \frac{n}{2^{n-1}}, \\ H_n &= 2\alpha^n = \frac{1}{2^{n-1}}, \\ O_n &= n\alpha^n = \frac{n}{2^n}, \end{aligned}$$

respectively.

Note that

$$\begin{aligned} G_n &= 2O_n, \\ G_n &= \frac{n}{2^{n-1}} = nH_n, \\ O_n &= \frac{n}{2^n} = \frac{n}{2}H_n. \end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 7. *Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized Oresme sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = 4 \times \frac{W_0 + (W_1 - W_0)x}{(x-2)^2}. \tag{2.7}$$

Proof. In Lemma 3, take $r = 1, s = -\frac{1}{4}$. □

The previous Lemma gives the following results as particular examples.

Corollary 8. *Generated functions of modified Oresme, Oresme-Lucas and Oresme numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} G_n x^n &= \frac{4x}{(x-2)^2}, \\ \sum_{n=0}^{\infty} H_n x^n &= -\frac{4}{x-2}, \\ \sum_{n=0}^{\infty} O_n x^n &= \frac{2x}{(x-2)^2}, \end{aligned}$$

respectively.

Proof. In Lemma 7, take $W_n = G_n$ with $G_0 = 0, G_1 = 1$, $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively. □

3 Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Oresme sequence $\{W_n\}_{n\geq 0}$.

Theorem 9 (Simson Formula of Generalized Oresme Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = \left(\frac{1}{4}\right)^n \begin{vmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{vmatrix}. \tag{3.1}$$

Proof. For a proof of Eq. (3.1), see Soykan [11], just take $s = -\frac{1}{4}$. □

The previous theorem gives the following results as particular examples.

Corollary 10. *For all integers n, modified Oresme, Oresme-Lucas and Oresme numbers are given as*

$$\begin{aligned} \begin{vmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{vmatrix} &= \frac{-1}{4^{n-1}}, \\ \begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix} &= 0, \\ \begin{vmatrix} O_{n+1} & O_n \\ O_n & O_{n-1} \end{vmatrix} &= \frac{-1}{4^n}, \end{aligned}$$

respectively.

4 Some Identities

In this section, we obtain some identities of generalized Oresme, modified Oresme, Oresme-Lucas and Oresme numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{G_n\}$.

Lemma 11. *The following equalities are true:*

$$
\begin{aligned}
W_n &= 16\left(2W_0-3W_1\right)G_{n+4}-4\left(5W_0-8W_1\right)G_{n+3}, \qquad (4.1)\\
W_n &= 4\left(3W_0-4W_1\right)G_{n+3}-4\left(2W_0-3W_1\right)G_{n+2},\\
W_n &= 4\left(W_0-W_1\right)G_{n+2}-\left(3W_0-4W_1\right)G_{n+1},\\
W_n &= W_0G_{n+1}-\left(W_0-W_1\right)G_n,\\
W_n &= W_1G_n-\frac{1}{4}W_0G_{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
\left(W_0-2W_1\right)^2G_n &= 64(W_0-3W_1)W_{n+4}-16(3W_0-8W_1)W_{n+3},\\
\left(W_0-2W_1\right)^2G_n &= 16(W_0-4W_1)W_{n+3}-16(W_0-3W_1)W_{n+2},\\
\left(W_0-2W_1\right)^2G_n &= -16W_1W_{n+2}-4(W_0-4W_1)W_{n+1},\\
\left(W_0-2W_1\right)^2G_n &= -4W_0W_{n+1}+4W_1W_n,\\
\left(W_0-2W_1\right)^2G_n &= -4(W_0-W_1)W_n+W_0W_{n-1}.
\end{aligned}
$$

Proof. Note that all the identities hold for all integers n. We prove (4.1). To show (4.1), writing

$$W_n = a\times G_{n+4}+b\times G_{n+3}$$

and solving the system of equations

$$
\begin{aligned}
W_0 &= a\times G_4+b\times G_3\\
W_1 &= a\times G_5+b\times G_4
\end{aligned}
$$

we find that $a=16\left(2W_0-3W_1\right),\ b=-4\left(5W_0-8W_1\right)$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{H_n\}$ and $\{W_n\}$.

Lemma 12. *The following equalities are true:*

$$\begin{aligned}
(W_0 - 2W_1)H_n &= -32W_{n+4} + 16W_{n+3}, \\
(W_0 - 2W_1)H_n &= -16W_{n+3} + 8W_{n+2}, \\
(W_0 - 2W_1)H_n &= -8W_{n+2} + 4W_{n+1}, \\
(W_0 - 2W_1)H_n &= -4W_{n+1} + 2W_n, \\
(W_0 - 2W_1)H_n &= -2W_n + W_{n-1}.
\end{aligned}$$

Now, we give a few basic relations between $\{W_n\}$ and $\{O_n\}$.

Lemma 13. *The following equalities are true:*

$$\begin{aligned}
W_n &= 32\left(2W_0 - 3W_1\right)O_{n+4} - 8\left(5W_0 - 8W_1\right)O_{n+3}, \\
W_n &= 8\left(3W_0 - 4W_1\right)O_{n+3} - 8\left(2W_0 - 3W_1\right)O_{n+2}, \\
W_n &= 8\left(W_0 - W_1\right)O_{n+2} - 2\left(3W_0 - 4W_1\right)O_{n+1}, \\
W_n &= 2W_0O_{n+1} - 2\left(W_0 - W_1\right)O_n, \\
W_n &= 2W_1O_n - \frac{1}{2}W_0O_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
\left(W_0 - 2W_1\right)^2 O_n &= 32(W_0 - 3W_1)W_{n+4} - 8(3W_0 - 8W_1)W_{n+3}, \\
\left(W_0 - 2W_1\right)^2 O_n &= 8(W_0 - 4W_1)W_{n+3} - 8(W_0 - 3W_1)W_{n+2}, \\
\left(W_0 - 2W_1\right)^2 O_n &= -8W_1W_{n+2} - 2(W_0 - 4W_1)W_{n+1}, \\
\left(W_0 - 2W_1\right)^2 O_n &= -2W_0W_{n+1} + 2W_1W_n, \\
\left(W_0 - 2W_1\right)^2 O_n &= -2(W_0 - W_1)W_n + \frac{1}{2}W_0W_{n-1}.
\end{aligned}$$

Now, we give a few basic relations between $\{G_n\}$, $\{H_n\}$ and $\{O_n\}$.

Lemma 14. *The following equalities are true:*

$$\begin{aligned} H_n &= 16G_{n+4} - 8G_{n+3}, \\ H_n &= 8G_{n+3} - 4G_{n+2}, \\ H_n &= 4G_{n+2} - 2G_{n+1}, \\ H_n &= 2G_{n+1} - G_n, \\ H_n &= G_n - \frac{1}{2}G_{n-1}, \end{aligned}$$

and

$$\begin{aligned} O_n &= -24G_{n+4} + 16G_{n+3}, \\ O_n &= -8G_{n+3} + 6G_{n+2}, \\ O_n &= -2G_{n+2} + 2G_{n+1}, \\ O_n &= \frac{1}{2}G_n, \end{aligned}$$

and

$$\begin{aligned} G_n &= -96O_{n+4} + 64O_{n+3}, \\ G_n &= -32O_{n+3} + 24O_{n+2}, \\ G_n &= -8O_{n+2} + 8O_{n+1}, \\ G_n &= 2O_n, \end{aligned}$$

and

$$\begin{aligned} H_n &= 32O_{n+4} - 16O_{n+3}, \\ H_n &= 16O_{n+3} - 8O_{n+2}, \\ H_n &= 8O_{n+2} - 4O_{n+1}, \\ H_n &= 4O_{n+1} - 2O_n, \\ H_n &= 2O_n - O_{n-1}, \end{aligned}$$

and

$$\begin{aligned} G_n &= \frac{n}{2^{n-1}} = nH_n, \\ O_n &= \frac{n}{2^n} = \frac{n}{2}H_n. \end{aligned}$$

We now present a few special identities for the generalized Oresme sequence $\{W_n\}$.

Theorem 15 (Catalan's identity of the generalized Oresme sequence)**.** *For all integers n and m, the following identity holds:*

$$W_{n+m}W_{n-m} - W_n^2 = -\frac{m^2}{2^{2n}}\left(W_0 - 2W_1\right)^2.$$

Proof. We use the identity

$$W_n = (W_0 + 2\left(W_1 - \frac{1}{2}W_0\right)n) \times \frac{1}{2^n}.$$

□

As special cases of the above theorem, we have the following corollary.

Corollary 16. *For all integers n and m, the following identities hold:*

(a) $G_{n+m}G_{n-m} - G_n^2 = -\frac{m^2}{2^{2n-2}}$.

(b) $H_{n+m}H_{n-m} - H_n^2 = 0$.

(c) $O_{n+m}O_{n-m} - O_n^2 = -\frac{m^2}{2^{2n}}$.

Note that for $m = 1$ in Catalan's identity of the generalized Oresme sequence, we get the Cassini identity for the generalized Oresme sequnce.

Theorem 17 (Cassini's identity of the generalized Oresme sequence)**.** *For all integers n, the following identity holds:*

$$W_{n+1}W_{n-1} - W_n^2 = -\frac{1}{2^{2n}}\left(W_0 - 2W_1\right)^2.$$

As special cases of the above theorem, we have the following corollary.

Corollary 18. *For all integers n, the following identities hold:*

(a) $G_{n+1}G_{n-1} - G_n^2 = -\frac{1}{2^{2n-2}}$.

(b) $H_{n+1}H_{n-1} - H_n^2 = 0.$

(c) $O_{n+1}O_{n-1} - O_n^2 = -\frac{1}{2^{2n}}.$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using

$$W_n = (W_0 + 2\left(W_1 - \frac{1}{2}W_0\right)n) \times \frac{1}{2^n}.$$

The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of generalized Oresme sequence $\{W_n\}$.

Theorem 19. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$W_{m+1}W_n - W_mW_{n+1} = -\frac{(m-n)}{2^{m+n+1}}\left(W_0 - 2W_1\right)^2.$$

(b) *(Gelin-Cesàro's identity)*

$$\begin{aligned} &W_{n+2}W_{n+1}W_{n-1}W_{n-2} - W_n^4 \\ &= -\frac{1}{2^{4n}}(4(5n^2-4)W_1^2+(5n^2-10n+1)W_0^2-4(5n^2-5n-4)W_1W_0)\left(W_0 - 2W_1\right)^2. \end{aligned}$$

(c) *(Melham's identity)*

$$W_{n+1}W_{n+2}W_{n+6}-W_{n+3}^3 = -\frac{1}{2^{3n+9}}(2(7n+15)W_1-(7n+8)W_0)\left(W_0 - 2W_1\right)^2.$$

Proof. Use the identity $W_n = (W_0 + 2\left(W_1 - \frac{1}{2}W_0\right)n) \times \frac{1}{2^n}$. □

As special cases of the above theorem, we have the following three corollaries. First one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified Oresme sequence $\{G_n\}$.

Corollary 20. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$G_{m+1}G_n - G_m G_{n+1} = -\frac{(m-n)}{2^{m+n-1}}.$$

(b) *(Gelin-Cesàro's identity)*

$$G_{n+2}G_{n+1}G_{n-1}G_{n-2} - G_n^4 = -\frac{\left(5n^2-4\right)}{2^{4n-4}}.$$

(c) *(Melham's identity)*

$$G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3 = -\frac{(7n+15)}{2^{3n+6}}.$$

Second one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of Oresme-Lucas sequence $\{H_n\}$.

Corollary 21. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$H_{m+1}H_n - H_m H_{n+1} = 0.$$

(b) *(Gelin-Cesàro's identity)*

$$H_{n+2}H_{n+1}H_{n-1}H_{n-2} - H_n^4 = 0.$$

(c) *(Melham's identity)*

$$H_{n+1}H_{n+2}H_{n+6} - H_{n+3}^3 = 0.$$

Third one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of Oresme sequence $\{O_n\}$.

Corollary 22. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$O_{m+1}O_n - O_mO_{n+1} = -\frac{(m-n)}{2^{m+n+1}}.$$

(b) *(Gelin-Cesàro's identity)*

$$O_{n+2}O_{n+1}O_{n-1}O_{n-2} - O_n^4 = -\frac{\left(5n^2-4\right)}{2^{4n}}.$$

(c) *(Melham's identity)*

$$O_{n+1}O_{n+2}O_{n+6} - O_{n+3}^3 = -\frac{(7n+15)}{2^{3n+9}}.$$

5 On the Recurrence Properties of Generalized Oresme Sequence

Taking $r = 1, s = -\frac{1}{4}$ in Theorem 1 (a) and (b), we obtain the following Proposition.

Proposition 23. *For $n \in \mathbb{Z}$, generalized Oresme numbers (the case $r = 1, s = -\frac{1}{4}$) have the following identity:*

$$\begin{aligned} W_{-n} &= (-1)^{n+1}\left(-\frac{1}{4}\right)^{-n}(W_n - H_nW_0) \\ &= \frac{(-1)^{n+1}\left(-\frac{1}{4}\right)^{-n}}{-W_1^2 - \frac{1}{4}W_0^2 + W_0W_1}((2W_1 - W_0)W_0W_{n+1} - (W_1^2 - \frac{1}{4}W_0^2)W_n). \end{aligned}$$

From the above Proposition, we have the following corollary which gives the connection between the special cases of generalized Oresme sequence at the positive index and the negative index: for modified Oresme, Oresme-Lucas and Oresme numbers: take $W_n = G_n$ with $G_0 = 0, G_1 = 1$, take $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively. Note that in this case $H_n = H_n$.

Corollary 24. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *modified Oresme sequence:*

$$G_{-n} = -4^n G_n = -n \times 2^{n+1}.$$

(b) *Oresme-Lucas sequence:*

$$H_{-n} = 4^n H_n = 2^{n+1}.$$

(c) *Oresme sequence:*

$$O_{-n} = -4^n O_n = -n \times 2^n.$$

6 The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}$ of Generalized Oresme Numbers

In this section, we present sum formulas of generalized Oresme numbers.

Theorem 25. *Let x be a real (or complex) number. For all integers m and j, for generalized Oresme numbers, we have the following sum formulas:*

(a) *If $2^{-2m}x^2 - xH_m + 1 \neq 0 \neq 0$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{\begin{array}{c}(2^{-2m}x - H_m)x^{n+1}W_{mn+j} \\ + 2^{-2m}x^{n+1}W_{mn+j-m} + W_j - 2^{-2m}xW_{j-m}\end{array}}{2^{-2m}x^2 - xH_m + 1}. \tag{6.1}$$

(b) *If $2^{-2m}x^2 - xH_m + 1 = u(x-a)(x-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{\begin{array}{c}(x(n+2)2^{-2m} - (n+1)H_m)x^n W_{j+mn} \\ + 2^{-2m}(n+1)x^n W_{mn+j-m} - 2^{-2m}W_{j-m}\end{array}}{2^{-2m+1}x - H_m}.$$

(c) *If $2^{-2m}x^2 - xH_m + 1 = u(x-c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{\begin{array}{c}(n+1)\left(2^{-2m}(n+2)x^n - nx^{n-1}H_m\right)W_{mn+j} \\ + n(n+1)2^{-2m}x^{n-1}W_{mn+j-m}\end{array}}{2^{-2m+1}}.$$

Proof. Take $r = 1, s = -\frac{1}{4}$ and $H_n = H_n$ in Theorem 2. $\square$

Note that (6.1) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{\begin{array}{r}(2^{-2m}x - H_m)x^{n+1}W_{mn+j} + 2^{-2m}x^{n+1}W_{mn+j-m} \\ + x(H_m - 2^{-2m}x)W_j - 2^{-2m}xW_{j-m}\end{array}}{2^{-2m}x^2 - xH_m + 1}.$$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 26. *For generalized Oresme numbers, we have the following sum formulas:*

(a) $(m = 1,\ j = 0)$

If $\frac{1}{4}(x-2)^2 \neq 0,$ *i.e.,* $x \neq 2,$ *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(x-4)\,x^{n+1}W_n + x^{n+1}W_{n-1} + 4W_0 + 4(W_1 - W_0)x}{(x-2)^2},$$

and

if $\frac{1}{4}(x-2)^2 = 0,$ *i.e.,* $x = 2,$ *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(n+1)((x-4)n + 2x)x^{n-1}W_n + n(n+1)x^{n-1}W_{n-1}}{2}.$$

(b) $(m = 2,\ j = 0)$

If $\frac{1}{16}(x-4)^2 \neq 0,$ *i.e.,* $x \neq 4,$ *then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(x-8)x^{n+1}W_{2n} + x^{n+1}W_{2n-2} + 16W_0 + 4\,(4W_1 - 3W_0)\,x}{(x-4)^2},$$

and

if $\frac{1}{16}(x-4)^2 = 0,$ *i.e.,* $x = 4,$ *then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(n+1)((x-8)n + 2x)x^{n-1}W_{2n} + n(n+1)x^{n-1}W_{2n-2}}{2}.$$

(c) $(m = 2,\ j = 1)$

If $\frac{1}{16}(x-4)^2 \neq 0,\ i.e.,\ x \neq 4,\ then$

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{(x-8)x^{n+1}W_{2n+1} + x^{n+1}W_{2n-1} + 16W_1 + 4(W_1 - W_0)x}{(x-4)^2},$$

and

if $\frac{1}{16}(x-4)^2 = 0,\ i.e.,\ x = 4,\ then$

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{(n+1)((x-8)n+2x)x^{n-1}W_{2n+1} + n(n+1)x^{n-1}W_{2n-1}}{2}.$$

(d) $(m = -1,\ j = 0)$

If $(2x-1)^2 \neq 0,\ i.e.,\ x \neq \frac{1}{2},\ then$

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{4x^{n+1}W_{-n+1} + 4(x-1)x^{n+1}W_{-n} + W_0 - 4xW_1}{(2x-1)^2},$$

and

if $(2x-1)^2 = 0,\ i.e.,\ x = \frac{1}{2},\ then$

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{n(n+1)x^{n-1}W_{-n+1} + (n+1)((x-1)n+2x)x^{n-1}W_{-n}}{2}.$$

(e) $(m = -2,\ j = 0)$

If $(4x-1)^2 \neq 0,\ i.e.,\ x \neq \frac{1}{4},\ then$

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{16x^{n+1}W_{-2n+2} + 8(2x-1)x^{n+1}W_{-2n} + W_0 - 4(4W_1 - W_0)x}{(4x-1)^2},$$

and

if $(4x-1)^2 = 0,\ i.e.,\ x = \frac{1}{4},\ then$

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{2n(n+1)x^{n-1}W_{-2n+2} + (n+1)((2x-1)n+4x)x^{n-1}W_{-2n}}{4}.$$

(f) $(m=-2,\ j=1)$

If $(4x-1)^2 \neq 0,\ i.e.,\ x \neq \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1} = \frac{16x^{n+1}W_{-2n+3} + 8\left(2x-1\right)x^{n+1}W_{-2n+1} + W_1 - 4\left(3W_1 - W_0\right)x}{\left(4x-1\right)^2},$$

and

if $(4x-1)^2 = 0,\ i.e.,\ x = \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1} = \frac{2n(n+1)x^{n-1}W_{-2n+3} + (n+1)((2x-1)n+4x)x^{n-1}W_{-2n+1}}{4}.$$

From the above proposition, we have the following corollary which gives sum formulas of modified Oresme numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1$).

Corollary 27. *For* $n \geq 0$, *modified Oresme numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $\frac{1}{4}(x-2)^2 \neq 0,\ i.e.,\ x \neq 2,$ *then*

$$\sum_{k=0}^{n} x^k G_k = \frac{\left(x-4\right)x^{n+1}G_n + x^{n+1}G_{n-1} + 4x}{\left(x-2\right)^2},$$

and

if $\frac{1}{4}(x-2)^2 = 0,\ i.e.,\ x = 2,$ *then*

$$\sum_{k=0}^{n} x^k G_k = \frac{(n+1)((x-4)n+2x)x^{n-1}G_n + n(n+1)x^{n-1}G_{n-1}}{2}.$$

(b) $(m=2,\ j=0)$

If $\frac{1}{16}(x-4)^2 \neq 0,\ i.e.,\ x \neq 4,$ *then*

$$\sum_{k=0}^{n} x^k G_{2k} = \frac{(x-8)x^{n+1}G_{2n} + x^{n+1}G_{2n-2} + 16x}{\left(x-4\right)^2},$$

and

if $\frac{1}{16}(x-4)^2 = 0$, *i.e.*, $x = 4$, *then*

$$\sum_{k=0}^{n} x^k G_{2k} = \frac{(n+1)((x-8)n+2x)x^{n-1}G_{2n} + n(n+1)x^{n-1}G_{2n-2}}{2}.$$

(c) $(m = 2,\ j = 1)$

If $\frac{1}{16}(x-4)^2 \neq 0$, *i.e.*, $x \neq 4$, *then*

$$\sum_{k=0}^{n} x^k G_{2k+1} = \frac{(x-8)x^{n+1}G_{2n+1} + x^{n+1}G_{2n-1} + 4x + 16}{(x-4)^2},$$

and

if $\frac{1}{16}(x-4)^2 = 0$, *i.e.*, $x = 4$, *then*

$$\sum_{k=0}^{n} x^k G_{2k+1} = \frac{(n+1)((x-8)n+2x)x^{n-1}G_{2n+1} + n(n+1)x^{n-1}G_{2n-1}}{2}.$$

(d) $(m = -1,\ j = 0)$

If $(2x-1)^2 \neq 0$, *i.e.*, $x \neq \frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k G_{-k} = \frac{4x^{n+1}G_{-n+1} + 4(x-1)x^{n+1}G_{-n} - 4x}{(2x-1)^2},$$

and

if $(2x-1)^2 = 0$, *i.e.*, $x = \frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k G_{-k} = \frac{n(n+1)x^{n-1}G_{-n+1} + (n+1)((x-1)n+2x)x^{n-1}G_{-n}}{2}.$$

(e) $(m = -2,\ j = 0)$

If $(4x-1)^2 \neq 0$, *i.e.*, $x \neq \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k G_{-2k} = \frac{16x^{n+1}G_{-2n+2} + 8(2x-1)x^{n+1}G_{-2n} - 16x}{(4x-1)^2},$$

and

if $(4x-1)^2 = 0,$ *i.e.,* $x = \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k G_{-2k} = \frac{2n(n+1)x^{n-1}G_{-2n+2} + (n+1)((2x-1)n+4x)x^{n-1}G_{-2n}}{4}.$$

(f) $(m = -2,\ j = 1)$

If $(4x-1)^2 \neq 0,$ *i.e.,* $x \neq \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k G_{-2k+1} = \frac{16x^{n+1}G_{-2n+3} + 8\left(2x-1\right)x^{n+1}G_{-2n+1} + 1 - 12x}{\left(4x-1\right)^2},$$

and

if $(4x-1)^2 = 0,$ *i.e.,* $x = \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k G_{-2k+1} = \frac{2n(n+1)x^{n-1}G_{-2n+3} + (n+1)((2x-1)n+4x)x^{n-1}G_{-2n+1}}{4}.$$

Taking $W_n = H_n$ with $H_0 = 2, H_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Oresme-Lucas numbers.

Corollary 28. *For* $n \geq 0,$ *Oresme-Lucas numbers have the following properties:*

(a) $(m = 1,\ j = 0)$

If $\frac{1}{4}\left(x-2\right)^2 \neq 0,$ *i.e.,* $x \neq 2,$ *then*

$$\sum_{k=0}^{n} x^k H_k = \frac{\left(x-4\right)x^{n+1}H_n + x^{n+1}H_{n-1} + 8 - 4x}{\left(x-2\right)^2},$$

and

if $\frac{1}{4}\left(x-2\right)^2 = 0,$ *i.e.,* $x = 2,$ *then*

$$\sum_{k=0}^{n} x^k H_k = \frac{(n+1)((x-4)n+2x)x^{n-1}H_n + n(n+1)x^{n-1}H_{n-1}}{2}.$$

(b) $(m=2,\ j=0)$

If $\frac{1}{16}(x-4)^2 \neq 0,$ *i.e.,* $x \neq 4,$ *then*

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{(x-8)x^{n+1}H_{2n} + x^{n+1}H_{2n-2} + 32 - 8x}{(x-4)^2},$$

and

if $\frac{1}{16}(x-4)^2 = 0,$ *i.e.,* $x = 4,$ *then*

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{(n+1)((x-8)n+2x)x^{n-1}H_{2n} + n(n+1)x^{n-1}H_{2n-2}}{2}.$$

(c) $(m=2,\ j=1)$

If $\frac{1}{16}(x-4)^2 \neq 0,$ *i.e.,* $x \neq 4,$ *then*

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{(x-8)x^{n+1}H_{2n+1} + x^{n+1}H_{2n-1} + 16 - 4x}{(x-4)^2},$$

and

if $\frac{1}{16}(x-4)^2 = 0,$ *i.e.,* $x = 4,$ *then*

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{(n+1)((x-8)n+2x)x^{n-1}H_{2n+1} + n(n+1)x^{n-1}H_{2n-1}}{2}.$$

(d) $(m=-1,\ j=0)$

If $(2x-1)^2 \neq 0,$ *i.e.,* $x \neq \frac{1}{2},$ *then*

$$\sum_{k=0}^{n} x^k H_{-k} = \frac{4x^{n+1}H_{-n+1} + 4(x-1)x^{n+1}H_{-n} + 2 - 4x}{(2x-1)^2},$$

and

if $(2x-1)^2 = 0,$ *i.e.,* $x = \frac{1}{2},$ *then*

$$\sum_{k=0}^{n} x^k H_{-k} = \frac{n(n+1)x^{n-1}H_{-n+1} + (n+1)((x-1)n+2x)x^{n-1}H_{-n}}{2}.$$

(e) $(m=-2,\ j=0)$

If $(4x-1)^2 \neq 0,\ i.e.,\ x \neq \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k} = \frac{16x^{n+1}H_{-2n+2} + 8\,(2x-1)\,x^{n+1}H_{-2n} + 2 - 8x}{(4x-1)^2},$$

and

if $(4x-1)^2 = 0,\ i.e.,\ x = \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k} = \frac{2n(n+1)x^{n-1}H_{-2n+2} + (n+1)((2x-1)n+4x)x^{n-1}H_{-2n}}{4}.$$

(f) $(m=-2,\ j=1)$

If $(4x-1)^2 \neq 0,\ i.e.,\ x \neq \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1} = \frac{16x^{n+1}H_{-2n+3} + 8\,(2x-1)\,x^{n+1}H_{-2n+1} + 1 - 4x}{(4x-1)^2},$$

and

if $(4x-1)^2 = 0,\ i.e.,\ x = \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1} = \frac{2n(n+1)x^{n-1}H_{-2n+3} + (n+1)((2x-1)n+4x)x^{n-1}H_{-2n+1}}{4}.$$

From the above proposition, we have the following corollary which gives sum formulas of Oresme numbers (take $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$).

Corollary 29. *For* $n \geq 0,$ *Oresme numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $\frac{1}{4}\,(x-2)^2 \neq 0,\ i.e.,\ x \neq 2,$ *then*

$$\sum_{k=0}^{n} x^k O_k = \frac{(x-4)\,x^{n+1}O_n + x^{n+1}O_{n-1} + 2x}{(x-2)^2},$$

and

if $\frac{1}{4}(x-2)^2 = 0$, *i.e.,* $x = 2$, *then*

$$\sum_{k=0}^{n} x^k O_k = \frac{(n+1)((x-4)n+2x)x^{n-1}O_n + n(n+1)x^{n-1}O_{n-1}}{2}.$$

(b) $(m = 2,\ j = 0)$

If $\frac{1}{16}(x-4)^2 \neq 0$, *i.e.,* $x \neq 4$, *then*

$$\sum_{k=0}^{n} x^k O_{2k} = \frac{(x-8)x^{n+1}O_{2n} + x^{n+1}O_{2n-2} + 8x}{(x-4)^2},$$

and

if $\frac{1}{16}(x-4)^2 = 0$, *i.e.,* $x = 4$, *then*

$$\sum_{k=0}^{n} x^k O_{2k} = \frac{(n+1)((x-8)n+2x)x^{n-1}O_{2n} + n(n+1)x^{n-1}O_{2n-2}}{2}.$$

(c) $(m = 2,\ j = 1)$

If $\frac{1}{16}(x-4)^2 \neq 0$, *i.e.,* $x \neq 4$, *then*

$$\sum_{k=0}^{n} x^k O_{2k+1} = \frac{(x-8)x^{n+1}O_{2n+1} + x^{n+1}O_{2n-1} + 2x + 8}{(x-4)^2},$$

and

if $\frac{1}{16}(x-4)^2 = 0$, *i.e.,* $x = 4$, *then*

$$\sum_{k=0}^{n} x^k O_{2k+1} = \frac{(n+1)((x-8)n+2x)x^{n-1}O_{2n+1} + n(n+1)x^{n-1}O_{2n-1}}{2}.$$

(d) $(m = -1,\ j = 0)$

If $(2x-1)^2 \neq 0$, *i.e.,* $x \neq \frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k O_{-k} = \frac{4x^{n+1}O_{-n+1} + 4(x-1)x^{n+1}O_{-n} - 2x}{(2x-1)^2},$$

and

if $(2x-1)^2 = 0$, *i.e.*, $x = \frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k O_{-k} = \frac{n(n+1)x^{n-1}O_{-n+1} + (n+1)((x-1)n+2x)x^{n-1}O_{-n}}{2}.$$

(e) $(m = -2,\ j = 0)$

If $(4x-1)^2 \neq 0$, *i.e.*, $x \neq \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k O_{-2k} = \frac{16x^{n+1}O_{-2n+2} + 8\left(2x-1\right)x^{n+1}O_{-2n} - 8x}{\left(4x-1\right)^2},$$

and

if $(4x-1)^2 = 0$, *i.e.*, $x = \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k O_{-2k} = \frac{2n(n+1)x^{n-1}O_{-2n+2} + (n+1)((2x-1)n+4x)x^{n-1}O_{-2n}}{4}.$$

(f) $(m = -2,\ j = 1)$

If $(4x-1)^2 \neq 0$, *i.e.*, $x \neq \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k O_{-2k+1} = \frac{32x^{n+1}O_{-2n+3} + 16\left(2x-1\right)x^{n+1}O_{-2n+1} + 1 - 12x}{2\left(4x-1\right)^2},$$

and

if $(4x-1)^2 = 0$, *i.e.*, $x = \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k O_{-2k+1} = \frac{2n(n+1)x^{n-1}O_{-2n+3} + (n+1)((2x-1)n+4x)x^{n-1}O_{-2n+1}}{4}.$$

Taking $x = 1$ in the last three corollaries we get the following corollary.

Corollary 30. *For* $n \geq 0$, *modified Oresme numbers, Oresme-Lucas numbers and Oresme numbers have the following properties:*

1.

(a) $\sum_{k=0}^{n} G_k = -3G_n + G_{n-1} + 4.$

(b) $\sum_{k=0}^{n} G_{2k} = \frac{1}{9}(-7G_{2n} + G_{2n-2} + 16).$

(c) $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{9}(-7G_{2n+1} + G_{2n-1} + 20).$

(d) $\sum_{k=0}^{n} G_{-k} = 4(G_{-n+1} - 1).$

(e) $\sum_{k=0}^{n} G_{-2k} = \frac{8}{9}(2G_{-2n+2} + G_{-2n} - 2).$

(f) $\sum_{k=0}^{n} G_{-2k+1} = \frac{1}{9}(16G_{-2n+3} + 8G_{-2n+1} - 11).$

2.

(a) $\sum_{k=0}^{n} H_k = -3H_n + H_{n-1} + 4.$

(b) $\sum_{k=0}^{n} H_{2k} = \frac{1}{9}(-7H_{2n} + H_{2n-2} + 24).$

(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{9}(-7H_{2n+1} + H_{2n-1} + 12).$

(d) $\sum_{k=0}^{n} H_{-k} = 2(2H_{-n+1} - 1).$

(e) $\sum_{k=0}^{n} H_{-2k} = \frac{2}{9}(8H_{-2n+2} + 4H_{-2n} - 3).$

(f) $\sum_{k=0}^{n} H_{-2k+1} = \frac{1}{9}(16H_{-2n+3} + 8H_{-2n+1} - 3).$

3.

(a) $\sum_{k=0}^{n} O_k = -3O_n + O_{n-1} + 2.$

(b) $\sum_{k=0}^{n} O_{2k} = \frac{1}{9}(-7O_{2n} + O_{2n-2} + 8).$

(c) $\sum_{k=0}^{n} O_{2k+1} = \frac{1}{9}(-7O_{2n+1} + O_{2n-1} + 10).$

(d) $\sum_{k=0}^{n} O_{-k} = 2(2O_{-n+1} - 1).$

(e) $\sum_{k=0}^{n} O_{-2k} = \frac{8}{9}(2O_{-2n+2} + O_{-2n} - 1).$

(f) $\sum_{k=0}^{n} O_{-2k+1} = \frac{1}{18}(32O_{-2n+3} + 16O_{-2n+1} - 11).$

7 Matrices Related with Generalized Oresme Numbers

We define the square matrix A of order 2 as:

$$A = \begin{pmatrix} 1 & -\frac{1}{4} \\ 1 & 0 \end{pmatrix}$$

such that $\det A = \frac{1}{4}$. Then, we have

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{4} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_n \\ W_{n-1} \end{pmatrix} \tag{7.1}$$

and

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{4} \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 \\ W_0 \end{pmatrix}.$$

If we take $W_n = G_n$ in (7.1) we have

$$\begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{4} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_n \\ G_{n-1} \end{pmatrix}. \tag{7.2}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & -\frac{1}{4}G_n \\ G_n & -\frac{1}{4}G_{n-1} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & -\frac{1}{4}W_n \\ W_n & -\frac{1}{4}W_{n-1} \end{pmatrix}.$$

Theorem 31. *For all integers m, n, we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. Take $r = 1, s = -\frac{1}{4}$ in Soykan [12, Theorem 5.1.]. □

Corollary 32. *For all integers n, we have the following formulas for the modified Oresme, Oresme-Lucas and Oresme numbers.*

(a) *Modified Oresme Numbers.*

$$A^n = \begin{pmatrix} 1 & -\frac{1}{4} \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & -\frac{1}{4}G_n \\ G_n & -\frac{1}{4}G_{n-1} \end{pmatrix}.$$

(b) *Oresme-Lucas Numbers.*

$$A^n = \begin{pmatrix} 1 & -\frac{1}{4} \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} (n+1)H_{n+1} & -\frac{1}{4}nH_n \\ nH_n & -\frac{1}{4}(n-1)H_{n-1} \end{pmatrix}.$$

(c) *Oresme Numbers.*

$$A^n = \begin{pmatrix} 1 & -\frac{1}{4} \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} 2O_{n+1} & -\frac{1}{2}O_n \\ 2O_n & -\frac{1}{2}O_{n-1} \end{pmatrix}.$$

Proof. **(a)** It is given in Theorem 31 (a).

(b) Note that, from Lemma 14, we have

$$G_n = nH_n.$$

Using the last equation and (a), we get required result.

(c) Note that, from Lemma 14, we have

$$G_n = 2O_n.$$

Using the last equation and (a), we get required result. □

Theorem 33. *For all integers m, n, we have*

$$W_{n+m} = W_nG_{m+1} - \frac{1}{4}W_{n-1}G_m \tag{7.3}$$

Proof. Take $r = 1, s = -\frac{1}{4}$ in Soykan [12, Theorem 5.2.]. □

By Lemma 11, we know that

$$(W_0 - 2W_1)^2 G_m = -4W_0W_{m+1} + 4W_1W_m,$$

so (7.3) can be written in the following form

$$(W_0 - 2W_1)^2 W_{n+m} = W_n(4(W_1-W_0)W_{m+1}+W_0W_m)+W_{n-1}(W_0W_{m+1}-W_1W_m).$$

Corollary 34. *For all integers m, n, we have*

$$\begin{aligned} G_{n+m} &= G_nG_{m+1} - \frac{1}{4}G_{n-1}G_m, \\ H_{n+m} &= H_nG_{m+1} - \frac{1}{4}H_{n-1}G_m, \\ O_{n+m} &= O_nG_{m+1} - \frac{1}{4}O_{n-1}G_m, \end{aligned}$$

and

$$O_{n+m} = 2O_nO_{m+1} - \frac{1}{2}O_{n-1}O_m.$$

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$f-$Biharmonic Curves in the Three-dimensional Para-Sasakian Space Forms

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Abstract

In this paper, we give some characterizations for proper $f-$biharmonic curves in the para-Bianchi-Cartan-Vranceanu space forms with 3-dimensional para-Sasakian structures.

1 Introduction

As a natural generalization of biharmonic curves, the concept of $f-$biharmonic curves was introduced by Lu in [4]. Since this paper, many authors studied f-biharmonic curves in several spaces: Ou considered $f-$biharmonic curves on a generic manifold and gave a characterization for them in $n-$dimensional space forms [6]. Guvenc and Ozgur studied $f-$biharmonic Legendre curves in Sasakian space forms [2]. Karaca and Ozgur investigated $f-$biharmonic curves in Sol spaces, Cartan Vranceanu three-dimensional spaces and homogenous contact three-manifolds [3]. Dua and Zhang examined $f-$biharmonic curves in Lorentz–Minkowski spaces [1].

On the other hand, in a very recent paper [5], Lee constructed the para-Bianchi-Cartan-Vranceanu model with 3-dimensional para-Sasakian structure and found the necessary and sufficient conditions for biharmonic Frenet curves.

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In this paper, we investigate $f-$biharmonic curves in this 3-dimensional para-Sasakian manifolds. We obtain some characterizations with respect to the special situations of curvature and torsion functions of these curves. Throughout the paper, all geometric objects (curves, manifolds, vector fields, functions etc.) are assumed to be smooth.

2 Preliminaries

2.1 Para-Sasakian manifolds

We recall fundamental ingredients of para-Sasakian manifolds from [5]. A $(2n+1)-$dimensional differentiable manifold M is said to be an almost paracontact manifold if it admits a (1,1)-tensor field φ, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = I - \eta \otimes \xi, \;\; \eta(\xi) = 1.$$

For an almost paracontact manifold M, we have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

If a $(2n+1)-$dimensional manifold M with almost paracontact structure (φ, ξ, η) admits a compatible pseudo-Riemannian metric such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

then we say M is an almost paracontact metric manifold with the paracontact metric structure (φ, ξ, η, g). Putting $Y = \xi$, we have

$$\eta(X) = g(X, \xi). \tag{2.2}$$

If the compatible pseudo-Riemannian metric g satisfies

$$d\eta(X, Y) = g(X, \varphi Y),$$

then η is a contact form on M, ξ the associated Reeb vector field, g an associated metric and $(M, \varphi, \xi, \eta, g)$ is called a paracontact metric manifold.

For a paracontact metric manifold M, an almost paracomplex structure J on $M \times \mathbb{R}$ is defined by

$$J(X, f\frac{d}{dt}) = (\varphi X + f\xi, \eta(X)\frac{d}{dt}),$$

where X is a vector field on M, t the coordinate of $\mathbb{R}$ and f a function of $M \times \mathbb{R}$. If the almost paracomplex structure J is integrable, then the paracontact metric manifold M is said to be normal or para-Sasakian.

Proposition 1. *[8] An almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is para-Sasakian if and only if*

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.3}$$

for any vector fields X, Y on M, where ∇ is Levi-Civita connection of g.

2.2 Frenet-Serret equations

Let $\gamma : I \to M$ be a unit speed curve in a three-dimensional Lorentzian manifold M such that γ' satisfies $g(\gamma', \gamma') = \varepsilon_1 = \pm 1$. The constant ε_1 is said to be the causal character of γ. A unit speed curve is called spacelike or timelike if its causal character is 1 or -1, respectively. A unit speed curve is called a Frenet curve if $g(\gamma'', \gamma'') \neq 0$. A Frenet curve has an orthonormal frame field $\{T = \gamma', N, B\}$ along γ. Then the Frenet-Serret equations are given by

$$\begin{aligned} \nabla_T T &= \varepsilon_2 \kappa N, \\ \nabla_T N &= -\varepsilon_1 \kappa T - \varepsilon_3 \tau B, \\ \nabla_T B &= \varepsilon_2 \tau N, \end{aligned}$$

where $\kappa = \left\|\nabla_{\gamma'}\gamma'\right\|$ is the geodesic curvature and τ is the geodesic torsion of γ. The vector fields T, N and B are called tangent vector field, principal normal vector field and binormal vector field of γ, respectively.

The constants ε_2 and ε_3 are defined by $g(N, N) = \varepsilon_2$ and $g(B, B) = \varepsilon_3$, and called second causal character and third causal character of γ, respectively. The equation $\varepsilon_1\varepsilon_2 = -\varepsilon_3$ holds.

A Frenet curve γ is a geodesic if and only if $\kappa = 0$.

Proposition 2. *Let $\{T, N, B\}$ are orthonormal frame field in a Lorentzian 3-manifold. Then,*

$$T \wedge_L N = \varepsilon_3 B,\ N \wedge_L B = \varepsilon_1 T,\ B \wedge_L T = \varepsilon_2 N.$$

2.3 $f-$Biharmonic maps

A map $\phi : (M_m, g) \to (N_n, h)$ between two pseudo-Riemannian manifolds is called harmonic if it is a critical point of the energy

$$E(\phi) = \frac{1}{2}\int_{\Omega} \|d\phi\|^2 \, dv_g,$$

where Ω is a compact domain of M_m. The tension field $\tau(\phi)$ of ϕ is defined by

$$\tau(\phi) = tr(\nabla^{\phi} d\phi) = \sum_{i=1}^{m} \varepsilon_i (\nabla^{\phi}_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)),$$

where ∇^{ϕ} and $\{e_i\}$ denote the induced connection by ϕ on the bundle $\phi^* TN_n$. A map ϕ is called harmonic if its tension field vanishes. The bienergy $E_2(\phi)$ of the map ϕ is defined by

$$E_2(\phi) = \frac{1}{2}\int_{\Omega} \|\tau(\phi)\|^2 \, dv_g,$$

and ϕ is called biharmonic if it is a critical point of the bienergy, where Ω is a compact domain of M_m. Clearly, all harmonic maps are biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps. The bitension field $\tau_2(\phi)$ of ϕ is defined by

$$\tau_2(\phi) = \sum_{i=1}^{m} \varepsilon_i ((\nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla_{e_i} e_i}) \tau(\phi) - R^N(\tau(\phi), d\phi(e_i)) d\phi(e_i)), \tag{2.4}$$

where R^N denotes the curvature tensor of N_n. A map ϕ is called biharmonic if its bitension field vanishes.

A map ϕ is called $f-$harmonic with a function $f : M \to \mathbb{R}$, if it is a critical point of the energy

$$E_f(\phi) = \frac{1}{2}\int_{\Omega} f \|d\phi\|^2 \, dv_g,$$

where Ω is a compact domain of M_m. The $f-$tension field $\tau_f(\phi)$ of ϕ is given by

$$\tau_f(\phi) = f\tau(\phi) + d\phi(gradf)$$

see [7]. The $f-$bitension field $\tau_{2,f}(\phi)$ of ϕ is defined by

$$\tau_{2,f}(\phi) = f\tau_2(\phi) + \Delta f\tau(\phi) + 2\nabla^{\phi}_{gradf}\tau(\phi). \tag{2.5}$$

A map ϕ is called $f-$biharmonic if its $f-$bitension field vanishes (see [1], [4]). Non-harmonic and non-biharmonic $f-$biharmonic curves are called proper $f-$biharmonic curves, and if f is constant, then an $f-$biharmonic curve turns into a biharmonic curve [4].

3 $f-$Biharmonic Curves in Para-Sasakian Space Forms

Lee introduced the concept of para-Bianchi-Cartan-Vranceanu model with 3-dimensional para-Sasakian structure in [5] as follows:

Consider the set

$$D = \{(x, y, z) \in \mathbb{R}^3 : 1 + \frac{c}{2}(x^2 + y^2) > 0\},$$

where c is a real number. Remark that if $c \geq 0$, then D is the whole $\mathbb{R}^3(x, y, z)$. On the region D, the contact form η is taken as

$$\eta = dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)}.$$

Then, the characteristic vector field of η is $\xi = \frac{\partial}{\partial z}$.

Next, the Lorentzian metric is equipped as

$$g_c = \frac{-dx^2 + dy^2}{\{1 + \frac{c}{2}(x^2 + y^2)\}^2} + (dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)})^2.$$

The Lorentzian orthonormal frame field (e_1, e_2, e_3) on (D, g_c) is given by

$$e_1 = \{1 + \frac{c}{2}(x^2 + y^2)\}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = \{1 + \frac{c}{2}(x^2 + y^2)\}\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z}.$$

Then the endomorphism field φ is given by

$$\varphi(e_1) = e_2,\ \varphi(e_2) = e_1,\ \varphi(e_3) = 0.$$

The Levi-Civita connection ∇ of (D, g_c) is described as

$$\begin{gathered}\nabla_{e_1}e_1 = -cye_2,\ \nabla_{e_1}e_2 = -cye_1 + e_3,\ \nabla_{e_1}e_3 = -e_2,\\ \nabla_{e_2}e_1 = -cxe_2 - e_3,\ \nabla_{e_2}e_2 = -cxe_1,\ \nabla_{e_2}e_3 = -e_1,\\ \nabla_{e_3}e_1 = -e_2, \nabla_{e_3}e_2 = -e_1, \nabla_{e_3}e_3 = 0.\end{gathered}$$

The contact form η on D fulfills

$$d\eta(X, Y) = g_c(X, \varphi Y),\ X, Y \in \chi(D).$$

Furthermore the structure $(g_c, \varphi, \xi, \eta)$ is para-Sasakian. The non-vanishing components of the curvature tensor R of (D, g_c) is given by

$$\begin{aligned}R(e_1, e_2)e_2 &= -\{3 + c^2(x^2 - y^2)\}e_1,\ R(e_1, e_3)e_3 = e_1,\\ R(e_2, e_1)e_1 &= \{3 + c^2(x^2 - y^2)\}e_2,\ R(e_2, e_3)e_3 = e_2,\\ R(e_3, e_1)e_1 &= -e_3,\ R(e_3, e_2)e_2 = e_3.\end{aligned}$$

For the sectional curvature K of (D, g_c), we have

$$K(e_2, e_3) = -1 = -K(e_3, e_1),$$

and

$$K(e_1, e_2) = R(e_1, e_2, e_1, e_2) = -\{3 + c^2(x^2 - y^2)\}.$$

So, (D, g_c) is of holomorphic sectional curvature $H = -\{3 + c^2(x^2 - y^2)\}$.

For the case $c = 0$, the holomorphic sectional curvature H equals -3, thus the space D becomes para-Sasakian space form. In the next, we will deal with the case $c = 0$.

Now, suppose that $\gamma : I \to (D, g_c)$ is a curve parametrized by arc-length and $\{T, N, B\}$ is an orthonormal frame field tangent to D along γ, where $T = T_1e_1 + T_2e_2 + T_3e_3$, $N = N_1e_1 + N_2e_2 + N_3e_3$ and $B = B_1e_1 + B_2e_2 + B_3e_3$.

The $f-$biharmonicity condition for curves on (D, g_c) is obtained in the following theorem.

Theorem 1. *Let $\gamma : I \to (D, g_c)$ be a curve parametrized by arc-length. Then γ is $f-$biharmonic if and only if the following relations are satisfied:*

$$\begin{gathered} 3\kappa\kappa' f + 2\kappa^2 f' = 0, \\ \kappa f'' + 2\kappa' f' + f[\kappa'' + \varepsilon_3\kappa^3 + \varepsilon_1\kappa\tau^2 + \kappa\varepsilon_2(\varepsilon_3 - 4\eta(B)^2)] = 0, \\ -2\kappa\tau f' - f(2\kappa'\tau + \kappa\tau') - 4\varepsilon_1\kappa f\eta(N)\eta(B) = 0. \end{gathered} \tag{3.1}$$

Proof. Let $\gamma = \gamma(s)$ be a curve parametrized by arc-length. We use formula (2.5). From [5], we have

$$\tau(\gamma) = \varepsilon_1\nabla_T T = -\varepsilon_3\kappa N, \tag{3.2}$$

$$\begin{aligned} R(T, N, T, N) &= \varepsilon_3 - 4B_3^2, \\ R(T, N, T, B) &= -4\varepsilon_1 N_3 B_3, \end{aligned} \tag{3.3}$$

$$\tau_2(\gamma) = 3\varepsilon_3\kappa\kappa' T + \varepsilon_2(\kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2))N + \varepsilon_1(2\kappa'\tau + \kappa\tau')B + \varepsilon_2\kappa R(T, N)T. \tag{3.4}$$

Moreover, from [1], we have

$$\begin{aligned} \nabla^\gamma_{grad f}\tau(\gamma) &= f'\nabla_T(\nabla_T T) = \varepsilon_2 f'[\kappa' N + \kappa(-\varepsilon_1\kappa T - \varepsilon_3\tau B)], \\ \Delta f\tau(\gamma) &= f''\nabla_T T = f''\varepsilon_2\kappa N. \end{aligned} \tag{3.5}$$

Therefore, combining the equations (3.2), (3.4) and (3.5), we obtain

$$\begin{aligned} \tau_{2,f}(\gamma) \;=\; & 3\varepsilon_3\kappa\kappa' f T + \varepsilon_2 f(\kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2))N + \varepsilon_1 f(2\kappa'\tau + \kappa\tau')B \\ & + \varepsilon_2 f\kappa R(T, N)T + \varepsilon_2\kappa f'' N + 2\varepsilon_2 f'[\kappa' N + \kappa(-\varepsilon_1\kappa T - \varepsilon_3\tau B)]. \end{aligned} \tag{3.6}$$

If we take inner product of equation (3.6) with T, N and B, respectively and use the equations (3.3), we get (3.1). □

Proposition 3. *Let $\gamma : I \to (D, g_c)$ be an $f-$biharmonic curve parametrized by arc-length. If κ is a non-zero constant, then γ is biharmonic.*

Proof. Under the assumption κ is a non-zero constant, from the first equation in (3.1), obviously we get $f' = 0$. So, γ is a biharmonic curve. □

Proposition 4. *Let $\gamma : I \to (D, g_c)$ be an $f-$biharmonic curve parametrized by arc-length. If τ is a non-zero constant and $\eta(N)\eta(B) = 0$, then γ is biharmonic.*

Proof. Under the assumption τ is a non-zero constant and $\eta(N)\eta(B) = 0$, using the first and third equations in (3.1), we get

$$\frac{\kappa'}{\kappa} = -\frac{2f'}{3f} \tag{3.7}$$

and

$$\tau(\frac{\kappa'}{\kappa} + \frac{f'}{f}) = 0. \tag{3.8}$$

Putting equation (3.7) in (3.8) shows that f is constant, therefore γ is a biharmonic curve. □

Proposition 5. *Let $\gamma : I \to (D, g_c)$ be an $f-$biharmonic curve parametrized by arc-length. If τ is a non-zero constant, then $f = e^{\int -\frac{6\varepsilon_1\eta(N)\eta(B)}{\tau}}$.*

Proof. Under the assumption τ is a non-zero constant, if we use the first and third equations in (3.1), we obtain

$$\frac{\kappa'}{\kappa} = -\frac{2f'}{3f} \tag{3.9}$$

and

$$-2\kappa\tau f' - 2f\kappa'\tau - 4\varepsilon_1\kappa f\eta(N)\eta(B) = 0. \tag{3.10}$$

Setting equation (3.9) in (3.10), we get the result. □

Proposition 6. *Let $\gamma : I \to (D, g_c)$ be a non-geodesic curve parametrized by arc-length and suppose that $\tau = 0$. In this case, γ is $f-$biharmonic if and only if the following equations are valid:*

$$f^2\kappa^3 = c_1^2, \tag{3.11}$$

$$(f\kappa)'' = -f\kappa(\varepsilon_3\kappa^2 + \varepsilon_2(\varepsilon_3 - 4\eta(B)^2)), \tag{3.12}$$

$$\eta(N)\eta(B) = 0, \tag{3.13}$$

where $c_1 \in \mathbb{R}$.

Proof. Under the assumption $\tau = 0$, if we use equations in (3.1) by integrating first equation, we deduce the results. □

Proposition 7. *Let $\gamma : I \to (D, g_c)$ be a non-geodesic curve parametrized by arc-length and suppose that τ and κ are non-constants. In this case, γ is $f-$biharmonic if and only if the following equations are valid:*

$$f^2\kappa^3 = c_1^2, \tag{3.14}$$

$$(f\kappa)'' = -f\kappa(\varepsilon_3\kappa^2 + \varepsilon_1\tau^2 + \varepsilon_2(\varepsilon_3 - 4\eta(B)^2)), \tag{3.15}$$

$$f^2\kappa^2\tau = e^{\int -\frac{4\varepsilon_1\eta(N)\eta(B)}{\tau}}, \tag{3.16}$$

where $c_1 \in \mathbb{R}$.

Proof. Under the assumption τ and κ are non-constants, if we use equations in (3.1) by integrating first and third equations, we obtain (3.14), (3.15) and (3.16). □

From the last two propositions, we can give the following theorem.

Theorem 2. *An arc-length parametrized curve $\gamma : I \to (D, g_c)$ is proper $f-$biharmonic if and only if one of the following situations is true:*

(i) $\tau = 0,\ f = c_1\kappa^{-3/2}$ and the curvature κ solves the equation below:

$$3(\kappa')^2 - 2\kappa\kappa'' = -4\kappa^2[\varepsilon_3\kappa^2 + \varepsilon_2(\varepsilon_3 - 4\eta(B)^2)].$$

(ii) $\tau \neq 0,\ \frac{\tau}{\kappa} = \frac{e^{\int -\frac{4\varepsilon_1\eta(N)\eta(B)}{\tau}}}{c_1^2},\ f = c_1\kappa^{-3/2}$ and the curvature κ solves the equation below:

$$3(\kappa')^2 - 2\kappa\kappa'' = -4\kappa^2[\varepsilon_3\kappa^2(1 - \varepsilon_2\frac{e^{\int -\frac{8\varepsilon_1\eta(N)\eta(B)}{\tau}}}{c_1^4}) + \varepsilon_2(\varepsilon_3 - 4\eta(B)^2)].$$

Proof. (i) The first equation of (3.1) gives

$$f = c_1\kappa^{-3/2}. \tag{3.17}$$

By replacing the above equation into (3.12), we obtain the result.

(ii) From the first equation of (3.1), we have

$$f = c_1\kappa^{-3/2}. \tag{3.18}$$

Setting the above equation in (3.16), we get

$$\frac{\tau}{\kappa} = \frac{e^{\int -\frac{4\varepsilon_1 \eta(N)\eta(B)}{\tau}}}{c_1^2}. \tag{3.19}$$

And finally putting equations (3.18) and (3.19) in (3.15), we obtain the result. □

Consequently, we can express the following corollary.

Corollary 1. *An arc-length parametrized $f-$biharmonic curve $\gamma : I \to (D, g_c)$ with constant geodesic curvature is biharmonic.*

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The Topp-Leone Weibull Distribution: Its Properties and Application

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Abstract

This paper presents a new generalization of the Topp-Leone distribution called the Topp-Leone Weibull Distribution (TLWD). Some of the mathematical properties of the proposed distribution are derived, and the maximum likelihood estimation method is adopted in estimating the parameters of the proposed distribution. An application of the proposed distribution alongside with some well-known distributions belonging to the Topp-Leone generated family of distributions, to a real lifetime data set reveals that the proposed distribution exhibits more flexibility in modeling lifetime data based on some comparison criteria such as maximized log-likelihood, Akaike Information Criterion $[AIC = 2k - 2\log(L)]$, Kolmogorov-Smirnov test statistic $(K-S)$ and Anderson Darling test statistic (A^*) and Crammer-Von Mises test statistic (W^*).

1. Introduction

Lifetime distributions are statistical models used for analyzing real life problems based on survival time. Many statistical distributions have been proposed to model lifetime data and the Topp-Leone distribution introduced by [14] is one of such distributions. In practice, most lifetime data sets encountered exhibits a bathtub hazard rate property and the one parameter Topp-Leone distribution happens to be the simplest distribution with such hazard rate property, but being a single parameter defined on a unit interval, its flexibility is limited in handling lifetime data sets.

Several generalizations of the distribution have been introduced to address the aforementioned drawback. These generalizations are found in the works of [1-5,9,10]. In this paper, we introduce a new generalization of the Topp-Leone distribution which serves as an alternative distribution among the Topp-Leone generated family of

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distributions. We shall call the proposed distribution, "Topp-Leone Weibull Distribution (TLWD)".

[14] proposed a J-shaped univariate distribution with cumulative distribution function defined by

$$F(x) = \left(\frac{x}{b}\right)^{\alpha}\left(2-\frac{x}{b}\right)^{\alpha}, \qquad 0 \le x \le b < \infty,\ 0 < \alpha < 1 \tag{1}$$

and the corresponding density function given by

$$f(x) = \frac{2\alpha}{b}\left(\frac{x}{b}\right)^{\alpha-1}\left(1-\frac{x}{b}\right)\left(2-\frac{x}{b}\right)^{\alpha-1}. \tag{2}$$

The survival function and the hazard rate function of Topp-Leone distribution are obtained using equations (1) and (2) as

$$S(x) = 1 - F(x) = 1 - \left(\frac{x}{b}\right)^{\alpha}\left(2-\frac{x}{b}\right)^{\alpha} \tag{3}$$

and

$$H(x) = \frac{F(x)}{1-F(x)} = \frac{\frac{2\alpha}{b}\left(\frac{x}{b}\right)^{\alpha-1}\left(1-\frac{x}{b}\right)}{\left\{\left(2-\frac{x}{b}\right)^{1-\alpha}-\left(\frac{x}{b}\right)^{\alpha}\left(2-\frac{x}{b}\right)\right\}}. \tag{4}$$

[11] improved on the mathematical properties of the Topp-Leone distribution by deriving higher moments for the distribution and a general usefulness of the distribution. Motivations for generalizing the Topp-Leone distribution arose after the work of [11]. One of such motivation is that, suppose a random variable x follow the Topp-Leone distribution then the random variable x can have either a finite support $(0 < x < b)$ or an infinite support $(0 < x < b < \infty)$.

Suppose we fix the parameter $b = 1$ in equations (1) and (2), then we obtain the cumulative distribution function and the density function of the one parameter Topp-Leone distribution defined on a unit interval respectively as

$$F(x) = x^{\alpha}(2-x)^{\alpha}, \qquad 0 < x < 1, \alpha > 0 \tag{5}$$

and

$$f(x) = 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}. \tag{6}$$

Figure 1 shows the graphical illustration of the density function and the hazard rate function of the Topp-Leone distribution for varying values of the shape parameter defined in the interval $0 < \alpha < 1$ and a fixed value of the scale parameter $b = 1$.

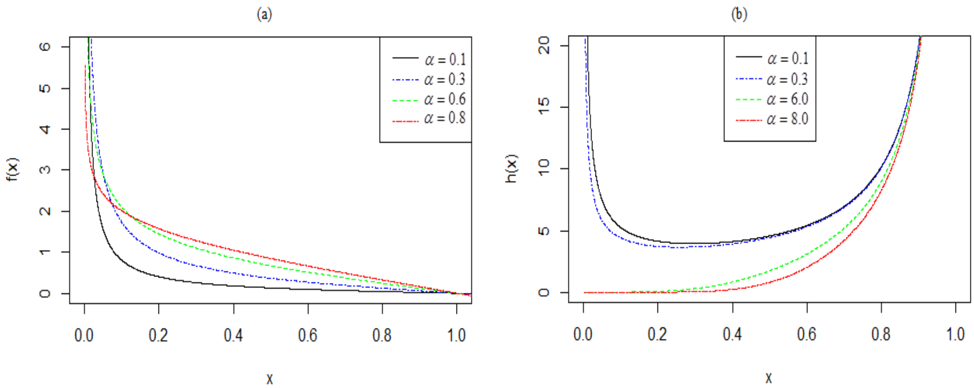

Figure 1: Density and hazard rate functions of the Topp-Leone distribution.

The plots in Figure 1(a), clearly shows that the density function of the TLD exhibits a reversed-J shape for different values of the shape parameter defined in the interval $0 < \alpha < 1$ and a fixed value of the scale parameter $b = 1$, while the plots in Figure 1(b) indicates that the hazard rate function of TLD exhibits a bathtub shape whenever $0 < \alpha < 1$ and a non-decreasing shape for $\alpha \geq 1$. The remaining sections of this paper are organized as follows: Section 2 presents some mathematical properties of the proposed distribution which include; the probability density function, cumulative distribution function, Hazard rate function, Survival function, Quantile function, Moments, Moment generating function, Renyi entropy, and distribution of ordered statistics. The model parameter estimation and simulation study on the maximum likelihood estimates of the proposed distribution are given in Section 3. Finally, in Section 4, we applied the proposed distribution to a real data set and compared its fit with the fit of some existing Topp-Leone generated family of distributions.

2. Mathematical Properties of the Proposed Distribution

2.1 The density and cumulative distribution functions of the proposed distribution

[13] introduced a new class of the Topp-Leone generated family of distributions with the cumulative distribution function defined as

$$F(x) = [G(x)]^{\alpha}[2 - G(x)]^{\alpha}, \tag{7}$$

and the corresponding density function given by

$$f(x) = 2\alpha g(x)\big(1 - G(x)\big)[G(x)]^{\alpha-1}[2 - G(x)]^{\alpha-1}. \tag{8}$$

Taking $G(x)$ as the distribution function of the Exponentiated exponential distribution developed by [8], the authors obtained the cumulative distribution function of the Topp-Leone Generalized Exponential Distribution as

$$F(x) = \left[1 - e^{-\lambda x}\right]^{\beta\alpha} \left[2 - \left(1 - e^{-\lambda x}\right)^{\beta}\right]^{\alpha}, \tag{9}$$

and the corresponding density function given by

$$f(x) = 2\alpha\beta\lambda e^{-\lambda x}\left[1 - e^{-\lambda x}\right]^{\beta\alpha - 1}\left(1 - \left(1 - e^{-\lambda x}\right)^{\beta}\right)$$

$$\times \left[2 - \left(1 - e^{-\lambda x}\right)^{\beta}\right]^{\alpha - 1}. \tag{10}$$

Using a similar approach of generating new distributions defined in equations (7) and (8), we assume the random variable X to follow the 2-parameter Weibull distribution with cumulative distribution function and probability density function respectively defined by

$$G(x) = 1 - e^{-\theta x^{\alpha}} \tag{11}$$

and

$$g(x) = \alpha\theta x^{\alpha - 1} e^{-\theta x^{\alpha}}. \tag{12}$$

Inserting equations (11) and (12) into (7) and (8), we define the cumulative distribution function and the probability distribution function of the Topp Leone Weibull distribution (TLWD) respectively as

$$F(x) = \left\{(1 - e^{-\theta x^{\alpha}})\left[(2 - (1 - e^{-\theta x^{\alpha}})\right]\right\}^{\lambda}$$

$$= \left\{1 - e^{-2\theta x^{\alpha}}\right\}^{\lambda} \tag{13}$$

and

$$f(x) = 2\lambda\alpha\theta x^{\alpha-1} e^{-\theta x^{\alpha}}\left[1 - \left(1 - e^{-\theta x^{\alpha}}\right)\right]\left[\left(1 - e^{-\theta x^{\alpha}}\right)\right]^{\lambda - 1}\left[2 - (1 - e^{-\theta x^{\alpha}})\right]^{\lambda - 1}$$

$$= 2\lambda\alpha\theta x^{\alpha-1} e^{-2\theta x^{\alpha}}\left(1 - e^{-2\theta x^{\alpha}}\right)^{\lambda - 1}. \tag{14}$$

The series representation of equation (14) can be obtain as

$$\left(1 - e^{-2\theta x^{\alpha}}\right)^{\lambda - 1} = \sum_{j=0}^{\infty} \binom{\lambda - 1}{j} (-1)^{j}\, e^{-2\theta x^{\alpha} j}$$

$$f(x) = 2\lambda\alpha\theta \sum_{j=0}^{\infty} \binom{\lambda - 1}{j} (-1)^{j}\, x^{\alpha - 1} e^{-2\theta x^{\alpha}(1+j)}. \tag{15}$$

The graphical plots of the probability density function are shown in Figure 2.

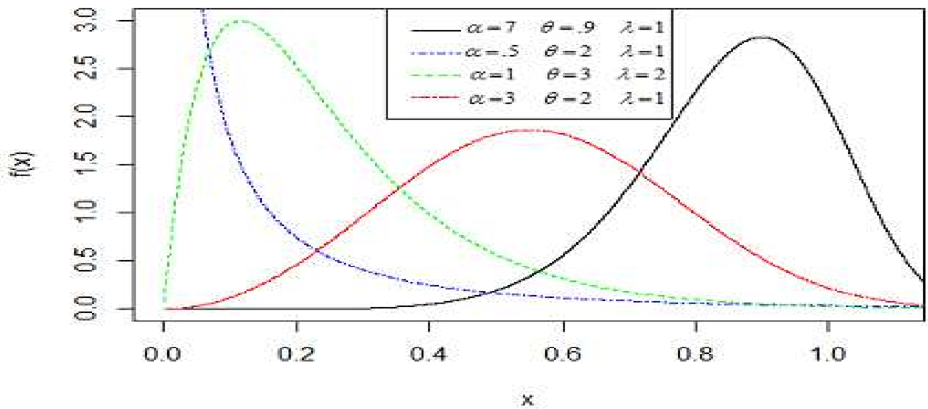

Figure 2: Density function of the TLWD for fixed value of the parameters.

The plots in Figure 2 clearly indicate that the density function of the TLWD exhibits a left-skewed, right-skewed, reversed J-shape and symmetric unimodal shapes.

2.2 Survival function and hazard function of the proposed distribution

Using equations (13) and (14), the survival function and the hazard rate function of the Topp Leone Weibull distribution are given by

$$S(x) = 1 - F(x)$$

$$= 1 - \left\{1 - e^{-2\theta x^{\alpha}}\right\}^{\lambda} \tag{16}$$

and

$$H(x) = \frac{f(x)}{1-F(x)}$$

$$= \frac{2\lambda\alpha\theta x^{\alpha-1} e^{-2\theta x^{\alpha}}}{\left\{\left(1-e^{-2\theta x^{\alpha}}\right)^{1-\lambda} - \left(1-e^{-2\theta x^{\alpha}}\right)\right\}}. \tag{17}$$

The graphical plots of the hazard rate function of the TLWD for different choice of the parameter value is shown in Figure 3.

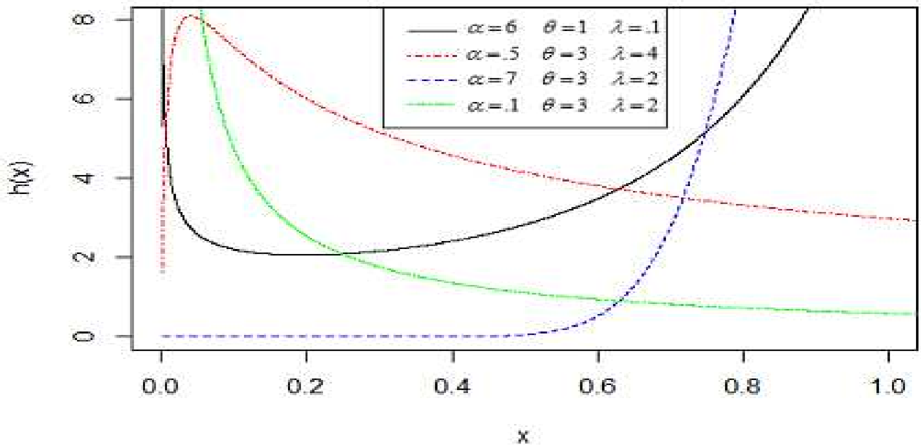

Figure 3: Hazard rate function of the TLWD for fixed value of the parameters.

2.3 Quantile function of the proposed distribution

The quantile function of a random variable x is obtained by solving the system equation of $F(x) = p$. Thus, given the cumulative distribution function $F(x)$ defined in equation (13), the p^{th} quantile function of the TLWD can be obtain as

$$\{1 - e^{-2\theta x^{\alpha}}\}^{\lambda} = p$$

$$1 - e^{-2\theta x^{\alpha}} = p^{1/\lambda}$$

$$e^{-2\theta x^{\alpha}} = 1 - p^{1/\lambda}$$

taking the natural logarithm of both sides we have

$$-2\theta x^{\alpha} = log\left(1 - p^{1/\lambda}\right)$$

$$x^{\alpha} = -\frac{1}{2\theta} log\left(1 - p^{1/\lambda}\right)$$

$$x = \left\{-\frac{1}{2\theta} log\left(1 - p^{1/\lambda}\right)\right\}^{1/\alpha}, \qquad 0 < p < 1. \tag{18}$$

The median of the TLWD can be obtained by substituting $p = \frac{1}{2}$ in equation (18) which yields,

$$median = \left\{-\frac{1}{2\theta} log\left(1 - (0.5)^{1/\lambda}\right)\right\}^{1/\alpha}. \tag{19}$$

2.4 Moments of the proposed distribution

Let X be a continuous random variable with density function $f(x)$, then the r^{th} moment about the origin of X is defined by

$$E[X^r] = \int_{-\infty}^{\infty} x^r f(x)dx \tag{20}$$

inserting the density function of the TLWD into equation (20), the r^{th} moment about the origin of the random variable X is defined by

$$E[X^r] \;=\; \int_0^{\infty} 2\lambda\alpha\theta x^{r+\alpha-1} e^{-2\theta x^{\alpha}} \left\{1 - e^{-2\theta x^{\alpha}}\right\}^{\lambda-1} dx \tag{21}$$

$$\left\{1 - e^{-2\theta x^{\alpha}}\right\}^{\lambda-1} = \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j e^{-2\theta x^{\alpha} j}$$

so that equation (21) now becomes,

$$E[X^r] \;=\; 2\lambda\alpha\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \int_0^{\infty} x^{r+\alpha-1} e^{-2\theta x^{\alpha}(1+j)} dx \tag{22}$$

evaluating the integral part of equation (22), we have

$$\int_0^{\infty} x^{r+\alpha-1} e^{-2\theta x^{\alpha}(1+j)} dx$$

let $y = 2\theta x^{\alpha}(1+j)$, which implies that, $x = \left[\frac{y}{2\theta(1+j)}\right]^{1/\alpha}$

so that, $dx = \frac{1}{\alpha 2\theta(1+j)} \left[\frac{y}{2\theta(1+j)}\right]^{1/\alpha - 1} dy$

$$\int_0^{\infty} \left\{\left[\frac{y}{2\theta(1+j)}\right]^{1/\alpha}\right\}^{r+\alpha-1} e^{-y} \frac{1}{\alpha 2\theta(1+j)} \left[\frac{y}{2\theta(1+j)}\right]^{1/\alpha - 1} dy$$

$$\int_0^{\infty} \frac{y^{r/\alpha + 1 - 1/\alpha + 1/\alpha - 1}}{\alpha[2\theta(1+j)]^{r/\alpha + 1 - 1/\alpha + 1/\alpha}} e^{-y} dy$$

$$\int_0^{\infty} \frac{y^{r/\alpha} e^{-y}}{\alpha[2\theta(1+j)]^{r/\alpha+1}} \, dy \;=\; \frac{\Gamma(r/\alpha + 1)}{\alpha[2\theta(1+j)]^{r/\alpha+1}} \, .$$

Thus, the r^{th} moment about the origin of the TLWD is given by

$$E[X^r] \;=\; 2\lambda\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{\Gamma(r/\alpha+1)}{[2\theta(1+j)]^{r/\alpha+1}} \, . \tag{23}$$

The first four raw moments of the TLWD are obtained from equation (23) as

$$\mu_1' = 2\lambda\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{\Gamma(1/\alpha+1)}{[2\theta(1+j)]^{1/\alpha+1}},$$

$$\mu_2' = 2\lambda\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{\Gamma(2/\alpha+1)}{[2\theta(1+j)]^{2/\alpha+1}},$$

$$\mu_3' = 2\lambda\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{\Gamma(3/\alpha+1)}{[2\theta(1+j)]^{3/\alpha+1}},$$

$$\mu_4' = 2\lambda\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{\Gamma(4/\alpha+1)}{[2\theta(1+j)]^{4/\alpha+1}}.$$

Furthermore, using the first four raw moments defined above, the measures of skewness (S_k) and kurtosis (K_s) are obtained as

$$S_k = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}} \quad \text{and} \quad K_s = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{(\mu_2' - \mu^2)^2}.$$

Tables 1 and 2 show the theoretical moments of the TLWD for different values of the parameters.

Table 1: Theoretical moments of TLWD for fixed value of the parameter $(\alpha = 2)$.

μ_r'	$(\theta = 2, \lambda = 3)$	$(\theta = 2, \lambda = 5)$	$(\theta = 4, \lambda = 3)$	$(\theta = 4, \lambda = 5)$
μ_1'	0.6452	0.7310	0.4562	0.5169
μ_2'	0.4583	0.5708	0.2292	0.2854
μ_3'	0.3542	0.4741	0.1252	0.1676
μ_4'	0.2951	0.4173	0.0738	0.1043
σ^2	0.0420	0.0364	0.0211	0.0182
S_k	0.4980	0.5143	0.4581	0.5078
K_s	3.2886	3.3859	3.5931	3.4248

Table 2: Theoretical moments of TLWD for fixed value of the parameter $(\alpha = 5)$.

μ_r'	$(\theta = 2, \lambda = 3)$	$(\theta = 2, \lambda = 5)$	$(\theta = 4, \lambda = 3)$	$(\theta = 4, \lambda = 5)$
μ_1'	0.8288	0.8750	0.7215	0.7617
μ_2'	0.6986	0.7740	0.5294	0.5866
μ_3'	0.5982	0.6921	0.3946	0.4566
μ_4'	0.5199	0.6254	0.2986	0.3592
σ^2	0.0117	0.0084	0.0088	0.0064
S_k	-0.1406	0.2517	-0.1393	0.0333
K_s	3.3840	0.9577	4.4313	5.0395

From Tables 1 and 2, we observed that the TLWD can be right skewed ($S_k > 0$), left skewed ($S_k < 0$) and approximately symmetric ($S_k \approx 0$). Also, at some fixed values of the parameters, the distribution can be leptokurtic ($K_s > 3$), platykurtic ($K_s < 3$) as well as mesokurtic ($K_s \approx 3$). This claim was clearly shown in Figure 2.

2.5 Moment generating function of the proposed distribution

Let X be a continuous random variable with density function $f(x)$, then the moment generating function of X is defined by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{24}$$

inserting the density function of the TLWD into equation (24), we obtain the moment generating function of the TLWD as

$$M_X(t) = \int_0^{\infty} e^{tx} 2\lambda\alpha\theta x^{\alpha-1} e^{-2\theta x^{\alpha}} \left\{1 - e^{-2\theta x^{\alpha}}\right\}^{\lambda-1} dx \tag{25}$$

using the series representation,

$$\left\{1 - e^{-2\theta x^{\alpha}}\right\}^{\lambda-1} = \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j e^{-2\theta x^{\alpha} j}$$

$$e^{tx} = \sum_{m=0}^{\infty} \frac{(tx)^m}{m!}$$

equation (25) becomes,

$$M_X(t) = 2\lambda\alpha\theta \sum_{j=0}^{\infty} \sum_{M=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{t^m}{m!} \int_0^{\infty} x^{m+\alpha-1} e^{-2\theta x^{\alpha}(1+j)} dx \tag{26}$$

evaluating the integral part of equation (26), we have

$$\int_0^{\infty} x^{m+\alpha-1} e^{-2\theta x^{\alpha}(1+j)} dx$$

$$= \int_0^{\infty} \left\{\left[\frac{y}{2\theta(1+j)}\right]^{1/\alpha}\right\}^{m+\alpha-1} e^{-y} \frac{1}{\alpha 2\theta(1+j)} \left[\frac{y}{2\theta(1+j)}\right]^{1/\alpha - 1} dy$$

$$= \int_0^{\infty} \frac{y^{m/\alpha + 1 - 1/\alpha + 1/\alpha - 1}}{\alpha[2\theta(1+j)]^{r/\alpha + 1 - 1/\alpha + 1/\alpha}} e^{-y} dy$$

$$= \int_0^{\infty} \frac{y^{m/\alpha} e^{-y}}{\alpha[2\theta(1+j)]^{m/\alpha+1}} dy$$

$$\int_0^{\infty} x^{m+\alpha-1} e^{-2\theta x^{\alpha}(1+j)} dx = \frac{\Gamma(m/\alpha + 1)}{\alpha[2\theta(1+j)]^{m/\alpha+1}} \tag{27}$$

upon substituting equation (27) into equation (26), the moment generating function of the

TLWD is given by

$$M_X(t) = 2\lambda\theta \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{t^m}{m!} \frac{\Gamma({}^{m}/_{\alpha}+1)}{[2\theta(1+j)]^{{}^{m}/_{\alpha}+1}}. \tag{28}$$

2.6 Renyi entropy of the proposed distribution

[12] defined an entropy of a random variable X as a measure of variation of uncertainty associated with the random variable X. The Renyi entropy of X with density function $f(x)$, is defined by,

$$\tau_R(\gamma) = \frac{1}{1-\gamma} log \int f^{\gamma}(x)dx\,, \qquad \gamma > 0, \quad \gamma \neq 1 \tag{29}$$

inserting the density function of the TLWD into equation (29), we obtain the Renyi entropy of the random variable X following the TLWD as

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log\left[(2\lambda\alpha\theta)^{\gamma} \int_0^{\infty} x^{(\alpha-1)\gamma}\, e^{-2\theta x^{\alpha}\gamma} \left\{1 - e^{-2\theta x^{\alpha}}\right\}^{(\lambda-1)\gamma} dx\right] \tag{30}$$

using series representation,

$$\left\{1 - e^{-2\theta x^{\alpha}}\right\}^{\lambda\gamma-\gamma} = \sum_{j=0}^{\infty} \binom{\lambda\gamma-\gamma}{j} (-1)^j\, e^{-2\theta x^{\alpha} j} \tag{31}$$

substituting equation (31) into equation (30), we have

$$= \frac{1}{1-\gamma} \log\left[(2\lambda\alpha\theta)^{\gamma} \sum_{j=0}^{\infty} \binom{\lambda\gamma-\gamma}{j} (-1)^j \int_0^{\infty} x^{(\alpha-1)\gamma} e^{-2\theta x^{\alpha}(\gamma+j)} dx\right] \tag{32}$$

again, evaluating the integral part of equation (32),

$$\int_0^{\infty} x^{(\alpha-1)\gamma} e^{-2\theta x^{\alpha}(\gamma+j)} dx$$

$$= \int_0^{\infty} \left\{\left[\frac{y}{2\theta(\gamma+j)}\right]^{{}^{1}/_{\alpha}}\right\}^{\alpha\gamma-\gamma} e^{-y} \frac{1}{\alpha 2\theta(\gamma+j)} \left[\frac{y}{2\theta(\gamma+j)}\right]^{{}^{1}/_{\alpha}-1} dy$$

$$= \int_0^{\infty} \frac{y^{\gamma - {}^{\gamma}/_{\alpha} + {}^{1}/_{\alpha} - 1}}{\alpha[2\theta(\gamma+j)]^{\gamma - {}^{\gamma}/_{\alpha} + {}^{1}/_{\alpha}}} e^{-y} dy$$

$$= \int_0^{\infty} \frac{y^{\gamma + {}^{(1-\gamma)}/_{\alpha} - 1}\, e^{-y}}{\alpha[2\theta(1+j)]^{\gamma + {}^{(1-\gamma)}/_{\alpha}}} \; dy$$

$$\int_0^{\infty} x^{(\alpha-1)\gamma} e^{-2\theta x^{\alpha}(\gamma+j)} dx = \frac{\Gamma\left(\gamma + {}^{(1-\gamma)}/_{\alpha}\right)}{\alpha[2\theta(\gamma+j)]^{\gamma + {}^{(1-\gamma)}/_{\alpha}}} \tag{33}$$

upon substituting equation (33) into equation (32), the Renyi entropy of the random

variable X following the TLWD is given by

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\left[(2\lambda\alpha\theta)^\gamma \sum_{j=0}^{\infty}\binom{\lambda\gamma-\gamma}{j}(-1)^j \frac{\Gamma\left(\gamma+{}^{(1-\gamma)}/_{\alpha}\right)}{\alpha[2\theta(\gamma+j)]^{\gamma+{}^{(1-\gamma)}/_{\alpha}}}\right]. \tag{34}$$

2.7 The distribution of the ordered statistics of the proposed distribution

Suppose that $Y_{1:n} < Y_{2:n} < \cdots < Y_{n:n}$ is the order statistics of a random sample generated from TLWD, then the probability density function of the k^{th} order statistics, say $X = Y_{n:n}$ is given by

$$h_i(x) = \frac{n!}{(n-k)!(k-1)!}[F(x)]^{k-1}[1-F(x)]^{n-k}f(x) \tag{35}$$

substituting the cumulative distribution function and the density function of TLWD defined in equations (13) and (14), into equation (35), we have

$$h_i(x) = \frac{2\lambda\alpha\theta n!}{(n-k)!(k-1)!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{n-k}{i}\binom{\lambda-1}{j}(-1)^{i+j}e^{-2\theta x^\alpha(1+j)}$$

$$\times\ x^{\alpha-1}\left[\left\{1-e^{-2\theta x^\alpha}\right\}^\lambda\right]^{i+k-1} \tag{36}$$

using the series representation,

$$\left\{1-e^{-2\theta x^\alpha}\right\}^{\lambda(i+k-1)} = \sum_{m=0}^{\infty}\binom{\lambda(i+k-1)}{m}(-1)^m e^{-2\theta x^\alpha m}$$

so that equation (36) becomes,

$$h_i(x) = \frac{2\lambda\alpha\theta n!}{(n-k)!(k-1)!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\binom{n-k}{i}\binom{\lambda-1}{j}\binom{\lambda(i+k-1)}{m}$$

$$\times\ (-1)^{i+j+m}x^{\alpha-1}e^{-2\theta x^\alpha(1+j+m)} \tag{37}$$

Hence, the s^{th} moment of the k^{th} order statistics from the TLWD is defined by

$$E(X_k^s) = \int_0^\infty x^s h_i(x)dx \tag{38}$$

$$= \frac{2\lambda\alpha\theta n!}{(n-k)!(k-1)!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\binom{n-k}{i}\binom{\lambda-1}{j}\binom{\lambda(i+k-1)}{m}$$

$$\times\ (-1)^{i+j+m}\int_0^\infty x^{s+\alpha-1}e^{-2\theta x^\alpha(1+j+m)} \tag{39}$$

evaluating the integral part of (39),

$$\int_0^\infty x^{s+\alpha-1}e^{-2\theta x^\alpha(1+j+m)}$$

let $y = 2\theta x^{\alpha}(1+j+m)$, which implies that, $x = \left[\frac{y}{2\theta(1+j+m)}\right]^{1/\alpha}$

so that, $dx = \frac{1}{\alpha 2\theta(1+j+m)}\left[\frac{y}{2\theta(1+j+m)}\right]^{1/\alpha - 1} dy$

$$\int_0^{\infty}\left\{\left[\frac{y}{2\theta(1+j+m)}\right]^{1/\alpha}\right\}^{s+\alpha-1} e^{-y}\frac{1}{\alpha 2\theta(1+j+m)}\left[\frac{y}{2\theta(1+j+m)}\right]^{1/\alpha - 1} dy$$

$$\int_0^{\infty}\frac{y^{S/\alpha+1-1/\alpha+1/\alpha-1}}{\alpha[2\theta(1+j+m)]^{S/\alpha+1-1/\alpha+1/\alpha}} e^{-y} dy$$

$$\int_0^{\infty}\frac{y^{S/\alpha}e^{-y}}{\alpha[2\theta(1+j+m)]^{S/\alpha+1}}\, dy = \frac{\Gamma(S/\alpha+1)}{\alpha[2\theta(1+j+m)]^{S/\alpha+1}}$$

equation (39) now becomes

$$E(X_k^s) = \frac{2\lambda\theta n!}{(n-k)!(k-1)!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{j=0}^{\infty}\binom{n-k}{i}\binom{\lambda-1}{j}\binom{\lambda(i+k-1)}{m}$$

$$\times(-1)^{i+j+m}\frac{\Gamma(S/\alpha+1)}{[2\theta(1+j+m)]^{S/\alpha+1}}. \tag{40}$$

3.0 Parameter Estimation of the Proposed Distribution

3.1 Maximum likelihood estimation

Let $(x_1, x_2, \dots, x_n)$ be random samples from the TLWD, then the log-likelihood function of the TLWD is defined by

$$\ell(x,\phi) = \sum_{i=1}^{n} log\left[2\lambda\alpha\theta x^{\alpha-1}e^{-2\theta x^{\alpha}}\left\{1 - e^{-2\theta x^{\alpha}}\right\}^{\lambda-1}\right], \qquad \phi = (\alpha,\theta,\lambda)^T$$

$$= nlog(2\lambda\alpha\theta) + (\alpha-1)\sum_{i=1}^{n} log(x_i) - 2\theta\sum_{i=1}^{n} x_i^{\alpha}$$

$$+(\lambda-1)\sum_{i=1}^{n} log\left(1-e^{-2\theta x^{\alpha}}\right). \tag{41}$$

On differentiating the log-likelihood function with respect to the parameters, we obtain the score function as,

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log(x_i) - 2\theta\sum_{i=1}^{n} x_i^{\alpha}\, log(x_i) + 2\theta(\lambda-1)\sum_{i=1}^{n}\frac{x_i^{\alpha} log(x_i) e^{-2\theta x_i^{\alpha}}}{\left\{1-e^{-2\theta x_i^{\alpha}}\right\}}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} log\left(1-e^{-2\theta x_i^{\alpha}}\right)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n} x_i^{\alpha} + 2(\lambda - 1)\sum_{i=1}^{n} \frac{x_i^{\alpha} e^{-2\theta x_i^{\alpha}}}{\left\{1 - e^{-2\theta x_i^{\alpha}}\right\}}.$$

The maximum likelihood estimator $\hat{\phi}$ of ϕ can be obtained by solving the system of equation $\frac{\partial \ell}{\partial \phi} = 0$. This equation cannot be solved using direct method since the system of equation is non-linear. However, an iterative method of solution such as the Newton Raphson method can be used. The Newton Raphson iterative scheme has the form:

$$\hat{\phi} = \phi_k - H^{-1}(\phi_k)U(\phi_k), \phi = (\alpha, \theta, \lambda)^T \tag{42}$$

where $U(\phi_k)$ is the score function and $H(\phi_k)$ is the Hessian matrix, which is the second derivative of the log-likelihood function. The Hessian matrix is defined by

$$H(\phi_k) = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \alpha \partial \alpha} & \frac{\partial^2 \ell}{\partial \alpha \partial \theta} & \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \theta \partial \alpha} & \frac{\partial^2 \ell}{\partial \theta \partial \theta} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ell}{\partial \lambda \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda \partial \lambda} \end{bmatrix}$$

where,

$$\frac{\partial^2 \ell}{\partial \alpha \partial \alpha} = -\frac{n}{\alpha^2} - 2\theta \sum_{i=1}^{n} x_i^{\alpha}\,[log(x_i)]^2 + 2\theta(\lambda - 1)\sum_{i=1}^{n} \frac{x_i^{\alpha}[log(x_i)]^2 e^{-2\theta x_i^{\alpha}}\left(1 - 2\theta x_i^{\alpha} - e^{-2\theta x_i^{\alpha}}\right)}{\left\{1 - e^{-2\theta x_i^{\alpha}}\right\}^2}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \theta} = \frac{\partial^2 \ell}{\partial \theta \partial \alpha} = -2\sum_{i=1}^{n} x_i^{\alpha}\, log(x_i) + 2(\lambda - 1)\sum_{i=1}^{n} \frac{x_i^{\alpha} log(x_i) e^{-2\theta x_i^{\alpha}}\left(1 - 2\theta x_i^{\alpha} - e^{-2\theta x_i^{\alpha}}\right)}{\left\{1 - e^{-2\theta x_i^{\alpha}}\right\}^2}$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \lambda} = \frac{\partial^2 \ell}{\partial \lambda \partial \theta} = 2\sum_{i=1}^{n} \frac{x_i^{\alpha} e^{-2\theta x_i^{\alpha}}}{\left\{1 - e^{-2\theta x_i^{\alpha}}\right\}}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = 2\theta \sum_{i=1}^{n} \frac{x_i^{\alpha} log(x_i) e^{-2\theta x_i^{\alpha}}}{\left\{1 - e^{-2\theta x_i^{\alpha}}\right\}}$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \lambda} = -\frac{n}{\lambda^2}$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \theta} = -\frac{n}{\theta^2} - 4(\lambda - 1)\sum_{i=1}^{n} \frac{x_i^{2\alpha} e^{-2\theta x_i^{\alpha}}}{\left\{1 - e^{-2\theta x_i^{\alpha}}\right\}^2}$$

3.2 Interval estimate

The asymptotic confidence intervals (CIs) for the parameters of TLWD $(\alpha, \theta, \lambda)$ are

obtained according to the asymptotic distribution of the maximum likelihood estimates of the parameters.

Suppose $\hat{\phi} = (\hat{\alpha}, \hat{\theta}, \hat{\lambda})$ is MLE of ϕ, then the estimators are approximately bi-variate normal with mean $(\alpha, \theta, \lambda)$ and the Fisher information matrix is given by:

$$I(\phi_k) = -E\big(H(\phi_k)\big). \tag{43}$$

The approximate $(1-\delta)100$ CIs for the parameters α, θ and λ respectively, are

$$\hat{\alpha} \pm Z_{\delta/2}\sqrt{var(\hat{\alpha})}\,, \hat{\theta} \pm Z_{\delta/2}\sqrt{var(\hat{\theta})} \text{ and } \hat{\lambda} \pm Z_{\delta/2}\sqrt{var(\hat{\lambda})}$$

where $var(\hat{\alpha})$, $var(\hat{\theta})$ and $var(\hat{\lambda})$ are the variance of α, θ and λ which are given by the diagonal elements of the variance-covariance matrix $I^{-1}(\phi_k)$ and $Z_{\delta/2}$ is the upper $(\delta/2)$ percentile of the standard normal distribution.

3.3 Simulation Study

In this section, we investigate the asymptotic behaviour of the maximum likelihood estimates of the parameters of the Topp-Leone Weibull distribution (TLWD) through a simulation study. A Monte Carlo simulation study is repeated 1000 times for different sample sizes n = 50, 75, 100, 200 and parameter values $(\alpha = 1.0, \theta = 0.3, \lambda = 0.5)$, $(\alpha = 1.0, \theta = 0.3, \lambda = 0.5)$ and $(\alpha = 1.0, \theta = 0.3, \lambda = 0.5)$. [6] suggested an algorithm for the simulation study employing the following steps.

1. Choose the value N (i.e. number of Monte Carlo simulation);

2. choose the values $\phi_0 = (\alpha_0, \theta_0, \lambda_0)$ corresponding to the parameters of the TLWD $(\alpha, \theta, \lambda)$;

3. generate a sample of size n from TLWD;

4. compute the maximum likelihood estimates $\hat{\phi}_0$ of ϕ_0;

5. repeat steps (3-4), N-times;

6. compute the:

(i) Average Bias $= \frac{1}{N}\sum_{i=1}^{N}\big(\widehat{\phi}_\iota - \phi_0\big)$,

(ii) Mean Square Error (MSE) $= \frac{1}{N}\sum_{i=1}^{N}\big(\widehat{\phi}_\iota - \phi_0\big)^2$ and

(iii) Coverage Probability of the 95% confidence interval of the estimates $\hat{\phi}_i$given by

$$CP(\phi_0) = \frac{1}{N}\sum_{i=1}^{N} I\left(\hat{\phi}_i - Z_{\delta/2}\sqrt{var(\hat{\phi})} < \phi_0 < \hat{\phi}_i + Z_{\delta/2}\sqrt{var(\hat{\phi})}\right)$$

where $I(\cdot)$ is an indicator function and $\sqrt{var(\hat{\phi})}$ is the standard error of the estimate ϕ_i. The coverage probability computes the proportion of times the confidence interval contains the true value of the parameter ϕ_0.

Table 3: Monte Carlo simulation results for average bias of the MLE.

Parameter	*N*	*Average Bias (α)*	*Average Bias (θ)*	*Average Bias (λ)*
$\alpha = 1.0$	50	0.2970	0.0314	0.0994
$\theta = 0.3$	75	0.1357	0.0293	0.0724
$\lambda = 0.5$	100	0.1364	0.0082	0.0362
	200	0.0507	0.0068	0.0157
$\alpha = 1.5$	50	0.4154	0.0341	0.1622
$\theta = 0.4$	75	0.2859	0.0141	0.0740
$\lambda = 0.6$	100	0.1714	0.0140	0.0540
	200	0.0721	0.0074	0.0242
$\alpha = 2.0$	50	0.5309	0.0403	0.2115
$\theta = 0.5$	75	0.2740	0.0087	0.0883
$\lambda = 0.8$	100	0.2340	-0.0114	0.0319
	200	0.0735	0.0104	0.0513

Table 4: Monte Carlo simulation results for mean square error (MSE) of the MLE.

Parameter	*N*	*MSE (α)*	*MSE (θ)*	*MSE(λ)*
$\alpha = 1.0$	50	0.6165	0.0727	0.2892
$\theta = 0.3$	75	0.2463	0.0422	0.1285
$\lambda = 0.5$	100	0.2105	0.0398	0.0916
	200	0.0598	0.0146	0.0271

$\alpha = 1.5$ $\theta = 0.4$ $\lambda = 0.6$	50	1.5398	0.0995	0.4345
	75	0.8736	0.0511	0.1777
	100	0.3915	0.0392	0.3917
	200	0.1208	0.0197	0.0477
$\alpha = 2.0$ $\theta = 0.5$ $\lambda = 0.8$	50	3.4213	0.0975	0.6521
	75	0.9752	0.0568	0.2400
	100	0.5449	0.0408	0.1602
	200	0.2143	0.0260	0.1023

Table 5: Monte Carlo simulation results for coverage probability of the 95% CIs of the MLE.

Parameter	*n*	*CP* (α)	*CP* (θ)	*CP* (λ)
$\alpha = 1.0$ $\theta = 0.3$ $\lambda = 0.5$	50	0.8200	0.8133	0.8333
	75	0.8800	0.8933	0.8933
	100	0.9200	0.8867	0.8967
	200	0.9300	0.9400	0.9467
$\alpha = 1.5$ $\theta = 0.4$ $\lambda = 0.6$	50	0.8667	0.8000	0.8167
	75	0.9200	0.8967	0.8867
	100	0.9433	0.9233	0.9033
	200	0.9500	0.9200	0.9133
$\alpha = 2.0$ $\theta = 0.5$ $\lambda = 0.8$	50	0.9167	0.8900	0.8700
	75	0.9567	0.9133	0.8900
	100	0.9700	0.9067	0.8800
	200	0.9300	0.9167	0.9233

Tables 3, 4 and 5 present the Monte Carlo simulation results for the average bias, mean square error and coverage probability of the 95% confidence interval of the parameter estimates of the TLWD at different choice of parameter values. Clearly from Table 3, we observe that the parameter α is positively biased, parameter λ is positively biased, while parameter θ could either be negatively or positively biased. Also from Table 4, as the sample size n increases, the values of the mean square error of the parameter estimates decreases (tends to zero). Finally, from Table 5, we observe that the coverage probabilities of the CIs are quite close to the nominal level of 95%.

4. Application of the Proposed Distribution

In this section, we fit the proposed distribution to a real data set alongside with some well-known lifetime distributions from the Topp-Leone Generated family of distributions with density functions given by;

(i) Topp-Leone Inverse Weibull Distribution (TLIWD)

$$f(x) = \frac{2\alpha\theta\lambda}{x^{\lambda+1}} e^{-\theta/x^{\lambda}} \left(1 - e^{-\theta/x^{\lambda}}\right) \left[1 - \left(1 - e^{-\theta/x^{\lambda}}\right)^2\right]^{\alpha-1};$$

(ii) Topp-Leone Bur XII Distribution (TLBXIID)

$$f(x) = 2\alpha\theta\lambda x^{\theta-1}\left(1 + x^{\theta}\right)^{-(2\alpha+1)} \left[1 - \left(1 + x^{\theta}\right)^{-2\alpha}\right]^{\lambda-1};$$

(iii) Topp-Leone Exponential Distribution (TLED)

$$f(x) = 2\alpha\lambda e^{-2\lambda x}\left(1 - e^{-2\lambda x}\right)^{\alpha-1};$$

(iv) Topp-Leone Linear Exponential Distribution (TLLED)

$$f(x) = 2\alpha(\theta + \lambda x)e^{-2\left(\theta x+\frac{\lambda x^2}{2}\right)} \left[1 - e^{-2\left(\theta x+\frac{\lambda x^2}{2}\right)}\right]^{\alpha-1};$$

(v) Topp-Leone Nadarajah-Haghighi Distribution (TLNHD)

$$f(x) = 2\alpha\theta\lambda(1 + \theta x)^{\lambda-1}e^{2\left(1-(1+\theta x)^{\lambda}\right)} \left[1 - e^{2\left(1-(1+\theta x)^{\lambda}\right)}\right]^{\alpha-1}.$$

The dataset consists of the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm reported in [7]. Table 6 presents the dataset.

Table 6: Tensile strength, measured in GPa, of 69 carbon fibers.

1.312	1.314	1.479	1.552	1.700	1.803	1.861
1.865	1.944	1.958	1.966	1.997	2.006	2.021
2.027	2.055	2.063	2.098	2.14	2.179	2.224
2.240	2.253	2.270	2.272	2.274	2.301	2.301
2.359	2.382	2.382	2.426	2.434	2.435	2.478
2.490	2.511	2.514	2.535	2.554	2.566	2.57
2.586	2.629	2.633	2.642	2.648	2.684	2.697
2.726	2.770	2.773	2.800	2.809	2.818	2.821
2.848	2.88	2.954	3.012	3.067	3.084	3.090
3.096	3.128	3.233	3.433	3.585	3.585	

The comparison criteria considered in this paper includes, the estimates of the parameters of the distribution, $Log - lik,$ Akaike Information Criterion $AIC = 2k - 2\log(L)]$, Kolmogorov-Smirnov Statistic $(K - S)$ and Anderson Darling test statistic (A^*) and Crammer-von Mises test statistic (W^*). Where, k is the number of estimated parameters and L is the value of the log-likelihood function evaluated at the parameter estimates.

Table 7: Summary statistics for the tensile strength of 69 carbon fibers dataset.

Models	*Estimates*	*Log-Lik*	*AIC*	*K-S* (*p-value*)	W^* (*p-value*)	A^* (*p-value*)
TLWD	$\alpha = 3.8590$ $\theta = 0.0194$ $\lambda = 2.0312$	-48.8598	103.7195	0.0395 (0.9999)	0.0154 (0.9996)	0.1435 (0.9990)
TLBXIID	$\alpha = 1.4510$ $\theta = 1.8205$ $\lambda = 110.9177$	-59.9967	125.9933	0.1216 (0.2591)	0.3164 (0.1215)	2.1117 (0.0799)
TLIWD	$\alpha = 0.5469$ $\theta = 34.8899$ $\lambda = 3.4115$	-58.0304	122.0608	0.1176 (0.2960)	0.2617 (0.1741)	1.7344 (0.1294)
TLED	$\alpha = 88.1467$ $\lambda = 1.0186$	-54.6201	113.2403	0.0950 (0.5617)	0.1601 (0.3608)	1.1210 (0.2994)
TLLED	$\alpha = 6.9393$ $\theta = -0.0826$ $\lambda = 0.4742$	-50.5881	107.1762	0.0660 (0.9240)	0.0627 (0.7985)	0.4406 (0.8072)
TLNHD	$\alpha = 21.4659$ $\theta = 0.0243$ $\lambda = 17.6504$	-50.4692	106.9383	0.0652 (0.9311)	0.0604 (0.8129)	0.4235 (0.8246)

Discussion of Result

The superiority of a model over another can be characterized by the model having the maximized loglikelihood and the least value in terms of *AIC, K-S* Statistic, Anderson

Darling test statistic (A^*) and Crammer-von Mises test statistic (W^*). Table 7 clearly reveal that the proposed distribution (TPGLD) having the maximized loglikelihood and the least values in terms of *AIC, K-S* Statistic, Anderson Darling test statistic (A^*) and Crammer-von Mises test statistic (W^*), demonstrates superiority over the Topp Leone Inverse Weibull Distribution (TLIWD), Topp Leone Bur XII Distribution (TLBXIID), Topp Leone Exponential Distribution (TLED), Topp Leone Linear Exponential Distribution (TLLED) and Topp Leone Nadarajah-Haghighi Distribution (TLNHD) in modeling the lifetime dataset under study. This claim was further supported by graphical illustration of the density and cumulative distribution fit of the distributions for the real lifetime dataset as displayed in Figure 4.

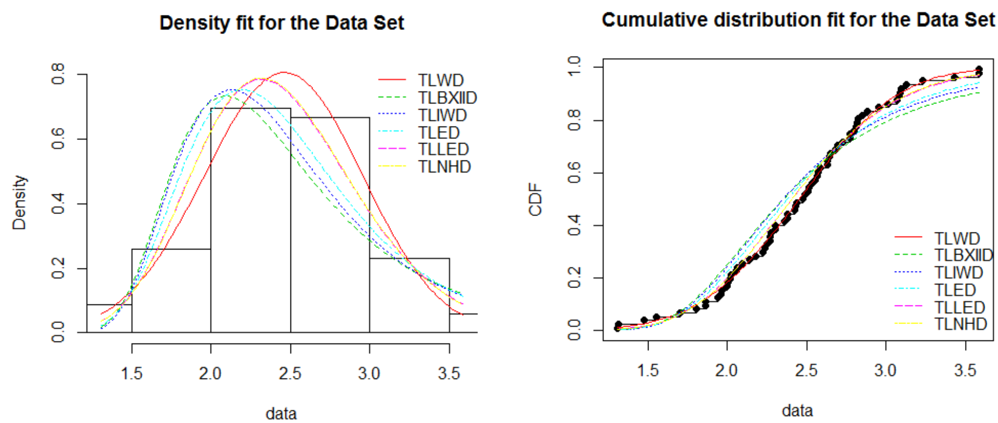

Figure 4: Density and cumulative distribution fit of the distributions for the dataset.

Conclusion

In this paper, we proposed a new member of the Topp-Leone generated family of distributions called the Topp-Leone Weibull Distribution (TLWD). The mathematical properties of the proposed distribution such as; the density function, cumulative distribution function, survival function, hazard rate function, moments and related measure, moment generating function, quantile function, Renyi entropy and the distribution of ordered statistics were derived. The method of maximum likelihood estimation was used in estimating the parameters of the proposed distribution.

Finally, an application of the proposed distribution to a real lifetime data set alongside with the Topp-Leone Bur XII distribution, Topp-Leone Inverse Weibull distribution,Topp-Leone Exponential distribution, Topp-Leone Linear Exponential distribution, and Topp-Leone Nadarajah-Haghighi distribution, reveals that the proposed

Topp-Leone Weibull distribution, demonstrates superiority and offers more flexibility in modeling the lifetime data set under study.

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New Families of Bi-Univalent Functions Governed by Gegenbauer Polynomials

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Abstract

The aim of this article is to initiating an exploration of the properties of bi-univalent functions related to Gegenbauer polynomials. To do so, we introduce a new families $\mathbb{T}_{\Sigma}(\gamma,\phi,\mu,,\eta,\theta,\lambda,t,\delta)$ and $\mathbb{S}_{\Sigma}(\sigma,\eta,\theta,\lambda,t,\delta)$ of holomorphic and bi-univalent functions. We derive estimates on the initial coefficients and solve the Fekete-Szegö problem of functions in these families.

1. Introduction

"In [20] Legendre studied orthogonal polynomials comprehensively. The importance of orthogonal polynomials for contemporary mathematics as well as for a wide range of their applications in physics and engineering, is beyond any doubt. It is well-known that these polynomials play an essential role in problems of the approximation theory. They occur in the theory of differential and integral equations as well as in mathematical statistics. Their applications in quantum mechanics, scattering theory, automatic control, signal analysis, and axially symmetric potential theory are also known [7,12]. In practical, the Gegenbauer polynomials is special case of orthogonal polynomials. They are representatively related with typically real functions T_R as discovered in [19]. Typically, real functions play an important role in the geometric function theory because of the relation $T_R = \overline{co}S_R$ and its role of estimating coefficient bounds, where S_R indicates the family of univalent functions in the unit disk with real coefficients and $\overline{co}S_R$ denotes the closed convex hull of S_R."

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On this subject in geometric function theory, the so-called Fekete-Szegö type inequalities (or problems) which estimate some upper bounds for $\left|a_3-\mu a_2^2\right|$ for holomorphic univalent functions. Its origin was in the disproof by Fekete and Szegö [16] conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity.

Let $\mathcal{A}$ stand for the collection of functions f have the type:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n\,, \tag{1.1}$$

which are holomorphic in the open unit disk $U=\{z\in\mathbb{C}\,:|z|<1\}$.

Further, symbolized by S the subfamily of $\mathcal{A}$ consisting the functions that are univalent in U.

According to the "Koebe One-Quarter Theorem" [13] each function f from S has an inverse f^{-1}, which fulfills

$$f^{-1}\big(f(z)\big)=z,\qquad (z\in U)$$

and

$$f\big(f^{-1}(w)\big)=w,\qquad \left(|w|<r_0(f),\, r_0(f)\geq\frac{1}{4}\right),$$

where

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

A function $f\in\mathcal{A}$ is named bi-univalent in U if together f and f^{-1} are univalent in U. Let Σ indicate the family of bi-univalent functions in U satisfying (1.1). Beginning with Srivastava et al. pioneering work [36] on the subject, the large number of works associated with the subject have been presented (see, for example [1,2,4,5,8,9,10,11,14, 17,18,21,22,25,28,29,30,31,32,33,34,35,37,38,39,40,41]). We see that the set Σ is not empty. We see that the functions

$$\frac{z}{1-z},\ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\ \text{ and }\ -\log(1-z)$$

are in the set Σ with the corresponding inverse functions

$$\frac{w}{1+w},\ \frac{e^{2w}-1}{e^{2w}+1}\ \text{ and }\ \frac{e^{w}-1}{e^{w}},$$

respectively. But the functions

$$z - \frac{z^2}{2} \text{ and } \frac{z}{1-z^2}.$$

are not a member of the set Σ.

The problem to find the bound of $|a_n|, (n = 3,4,\dots)$ of functions $f \in \Sigma$ is still an open problem.

The fundamental distributions like, the Pascal, the Binomial, the Poisson, the Logarithmic, the Borel have been partially considered in the "Geometric Function Theory" from a theoretical point of view (see for example [6,15,24,26,43]).

We say that the discrete random variable x have a beta negative binomial distribution, if it has the values $0, 1, 2, 3, \dots$ with the probabilities $\frac{\beta(\eta+\theta,\lambda)}{\beta(\eta,\lambda)}, \theta\frac{\beta(\eta+\theta,\lambda+1)}{\beta(\eta,\lambda)}, \frac{1}{2}\theta(\theta+1)\frac{\beta(\eta+\theta,\lambda+2)}{\beta(\eta,\lambda)}, \dots$, respectively, where η, θ, λ are named the parameters.

$$\operatorname{Prob}(x=\tau) = \binom{\theta+\tau-1}{\tau}\frac{\beta(\eta+\theta,\lambda+\tau)}{\beta(\eta,\lambda)} = \frac{\Gamma(\theta+\tau)}{\tau!\ \Gamma(\theta)}\frac{\Gamma(\eta+\theta)\Gamma(\lambda+\tau)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+\tau)\Gamma(\eta)\Gamma(\lambda)}$$

$$= \frac{(\eta)_\theta(\theta)_\tau(\lambda)_\tau}{(\eta+\lambda)_\theta(\theta+\eta+\lambda)_\tau\,\tau!}, \quad \tau = 0,1,2,3,\dots,$$

where $(\alpha)_n$ is the Pochhammer symbol defined by

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1 & (n=0), \\ \alpha(\alpha+1)\dots(\alpha+n-1) & (n \in \mathbb{N}). \end{cases}$$

Recently, Wanas and Al-Ziadi [42] studied the following power series whose coefficients are probabilities of the beta negative binomial distribution:

$$\mathfrak{X}_{\eta,\lambda}^{\theta}(z) = z + \sum_{n=2}^{\infty}\frac{(\eta)_\theta(\theta)_{n-1}(\lambda)_{n-1}}{(\eta+\lambda)_\theta(\theta+\eta+\lambda)_{n-1}(n-1)!}z^n, \quad z \in U,$$

where $\eta, \lambda, \theta > 0$. We see that, by making use of ratio test we deduce that the radius of convergence of the above power series is infinity.

Now, we consider the linear operator $\mathfrak{B}_{\eta,\lambda}^{\theta} : \mathcal{A} \longrightarrow \mathcal{A}$ which is defined as follows:

$$\mathfrak{B}_{\eta,\lambda}^{\theta}f(z) = \mathfrak{X}_{\eta,\lambda}^{\theta}(z) * f(z) = z + \sum_{n=2}^{\infty}\frac{(\eta)_\theta(\theta)_{n-1}(\lambda)_{n-1}}{(\eta+\lambda)_\theta(\theta+\eta+\lambda)_{n-1}(n-1)!}a_n z^n, \quad z \in U,$$

where "$*$" indicate the convolution of two series.

"For the functions f and g be holomorphic in U. We say that the function f is said to be subordinate to g, if there exists a Schwarz function w holomorphic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z) (z \in U)$. It is well known that (see [23]), if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$."

Recently, Amourah [3] studied the generating function of Gegenbauer polynomials $H_\delta(z,t)$ that are given by the following recurrence relation:

$$H_\delta(z,t) = \frac{1}{(1-2tz+z^2)^\delta},$$

where $\delta \in \mathbb{R} \setminus \{0\}$, $t \in [-1,1]$ and $z \in U$. For fixed t, the function H_δ is holomorphic in U, so it may be expanded in a Taylor-Maclaurin series as note that if $t = \cos\beta$, where $\beta \in (-\frac{\pi}{3}, \frac{\pi}{3})$, then

$$H_\delta(z,t) = \frac{1}{(1-2tz+z^2)^\delta} = \sum_{n=0}^{\infty} \mathcal{G}_n^\delta(t) z^n,$$

where $\mathcal{G}_n^\delta(t)$ is Gegenbauer polynomial of degree n.

Clearly, H_δ generates nothing when $\delta = 0$. Thus, the generating function of the Gegenbauer polynomial is set to be

$$H_0(z,t) = 1 - \log(1-2tz+z^2) = \sum_{n=0}^{\infty} \mathcal{G}_n^0(t) z^n.$$

Furthermore, it is worth to mention that a normalization of δ to be greater than $-\frac{1}{2}$ is desirable [12,27]. Also, Gegenbauer polynomials can be introduced by the following recurrence relations:

$$\mathcal{G}_n^\delta(t) = \frac{1}{2}\left[2t(n+\delta-1)\mathcal{G}_{n-1}^\delta(t) - (n+2\delta-2)\mathcal{G}_{n-1}^\delta(t)\right],$$

with the initial values

$$\mathcal{G}_0^\delta(t) = 1, \ \ \mathcal{G}_1^\delta(t) = 2\delta t \ \text{ and } \ \mathcal{G}_2^\delta(t) = 2\delta(\delta+1)t^2 - \delta\,. \tag{1.3}$$

Remark 1.1. Choosing the special values of δ, the Gegenbauer polynomial $\mathcal{G}_n^\delta(t)$ reduces to the following well-known polynomials:

1) Taking $\delta = 1$, we have the Chebyshev Polynomials.

2) Taking $\delta = \frac{1}{2}$, we obtain the Legendre Polynomials.

2. Main Results

This section start with defining the families $\mathbb{T}_{\Sigma}(\gamma, \phi, \mu, , \eta, \theta, \lambda, t, \delta)$ and $\mathbb{S}_{\Sigma}(\sigma, \eta, \theta, \lambda, t, \delta)$ as follows:

Definition 2.1. For $0 \le \gamma \le 1, 0 \le \phi \le 1, 0 \le \mu \le 1$, $t \in \left(\frac{1}{2}, 1\right]$ and δ is a nonzero real constant, a function $f \in \Sigma$ is called in the family $\mathbb{T}_{\Sigma}(\gamma, \phi, \mu, \eta, \theta, \lambda, t, \delta)$ if it fulfills the subordinations:

$$\left(\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)}\right)^{\gamma} \left[(1-\mu)\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)} + \mu\left(1 + \frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'}\right)\right]^{\phi} \prec \frac{1}{(1-2tz+z^2)^{\delta}}$$

and

$$\left(\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)}\right)^{\gamma} \left[(1-\mu)\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)} + \mu\left(1 + \frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'}\right)\right]^{\phi} \prec \frac{1}{(1-2tw+w^2)^{\delta}},$$

where the function $g = f^{-1}$ is given by (1.2).

Definition 2.2. For $0 \le \sigma \le 1$, $t \in \left(\frac{1}{2}, 1\right]$ and δ is a nonzero real constant, a function $f \in \Sigma$ is called in the family $\mathbb{S}_{\Sigma}(\sigma, \eta, \theta, \lambda, t, \delta)$ if it fulfills the subordinations:

$$\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)' + (2\sigma+1)z^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'' + \sigma z^3\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'''}{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)' + \sigma z^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)''} \prec \frac{1}{(1-2tz+z^2)^{\delta}}$$

and

$$\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)' + (2\sigma+1)w^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'' + \sigma w^3\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'''}{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)' + \sigma w^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)''} \prec \frac{1}{(1-2tw+w^2)^{\delta}},$$

where the function $g = f^{-1}$ is given by (1.2).

In particular, if we choose $\phi = 0$ and $\gamma = 1$ in Definition 2.1, the family $\mathbb{T}_{\Sigma}(\gamma, \phi, \mu, , \eta, \theta, \lambda, t, \delta)$ reduces to the family $\mathfrak{T}_{\Sigma}(\eta, \theta, \lambda, t, \delta)$ of bi-starlike functions

which fulfills the conditions:

$$\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)} \prec \frac{1}{(1-2tz+z^2)^{\delta}}$$

and

$$\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)} \prec \frac{1}{(1-2tw+w^2)^{\delta}},$$

where the function $g = f^{-1}$ is given by (1.2).

If we choose $\sigma = 0$ in Definition 2.2, the family $\mathbb{S}_{\Sigma}(\sigma,\eta,\theta,\lambda,t,\delta)$ reduces to the family $\mathfrak{H}_{\Sigma}(\eta,\theta,\lambda,t,\delta)$ of bi-convex functions which fulfills the conditions:

$$1+\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'} \prec \frac{1}{(1-2tz+z^2)^{\delta}}$$

and

$$1+\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'} \prec \frac{1}{(1-2tw+w^2)^{\delta}},$$

where the function $g = f^{-1}$ is given by (1.2).

Theorem 2.1. *For $0 \le \gamma \le 1, 0 \le \phi \le 1, 0 \le \mu \le 1,\ t \in (\frac{1}{2},1]$ and δ is a nonzero real constant, let $f \in \mathcal{A}$ be in the family $\mathbb{T}_{\Sigma}(\gamma,\phi,\mu,\eta,\theta,\lambda,t,\delta)$. Then*

$$|a_2| \le \frac{2|\delta|t\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\sqrt{2|\delta|t}}{\sqrt{\left|\begin{array}{c}\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -2\left[\begin{array}{c}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2\end{array}\right|}}$$

and

$$|a_3| \le \frac{4\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\delta^2t^2}{\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}$$

$$+\frac{2\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)|\delta|t}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)},$$

where

$$\Omega(\gamma,\phi,\mu,\eta,\theta,\lambda)=\frac{2(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\lambda+2)}{\Gamma(\eta+\theta+\lambda+2)}$$

$$+\frac{\theta\Gamma(\eta+\theta)\Gamma^2(\lambda+1)\Gamma(\eta+\lambda)\big[\gamma(\gamma-1)+\phi(\mu+1)\big(2\gamma+(\phi-1)(\mu+1)\big)-2\big(\gamma+\phi(3\mu+1)\big)\big]}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}. \quad (2.1)$$

Proof. Let $f\in\mathbb{T}_\Sigma(\gamma,\phi,\mu,\eta,\theta,\lambda,t,\delta)$. Then there exists two holomorphic functions $u,v\colon U\longrightarrow U$ given by

$$u(z)=u_1z+u_2z^2+u_3z^3+\cdots \qquad (z\in U) \quad (2.2)$$

and

$$v(w)=v_1w+v_2w^2+v_3w^3+\cdots \qquad (w\in U), \quad (2.3)$$

with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$, $z,w\in U$ such that

$$\left(\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)}\right)^{\gamma}\left[(1-\mu)\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)}+\mu\left(1+\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}\right)\right]^{\phi}$$

$$=1+\mathcal{G}_1^{\delta}(t)u(z)+\mathcal{G}_2^{\delta}(t)u^2(z)+\cdots \quad (2.4)$$

and

$$\left(\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)}\right)^{\gamma}\left[(1-\mu)\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)}+\mu\left(1+\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}\right)\right]^{\phi}$$

$$=1+\mathcal{G}_1^{\delta}(t)v(w)+\mathcal{G}_2^{\delta}(t)v^2(w)+\cdots\ . \quad (2.5)$$

Combining (2.2), (2.3), (2.4) and (2.5), we obtain

$$\left(\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)}\right)^{\gamma}\left[(1-\mu)\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)}+\mu\left(1+\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}\right)\right]^{\phi}$$

$$=1+\mathcal{G}_1^{\delta}(t)u_1z+\left[\mathcal{G}_1^{\delta}(t)u_2+\mathcal{G}_2^{\delta}(t)u_1^2\right]z^2+\cdots \quad (2.6)$$

and

$$\left(\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)}\right)^{\gamma}\left[(1-\mu)\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)}+\mu\left(1+\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}\right)\right]^{\phi}$$

$$= 1+\mathcal{G}_1^{\delta}(t)v_1w+\left[\mathcal{G}_1^{\delta}(t)v_2+\mathcal{G}_2^{\delta}(t)v_1^2\right]w^2+\cdots \ . \tag{2.7}$$

It is quite well-known that if $|u(z)|<1$ and $|v(w)|<1$, $z,w\in U$, then

$$|u_i|\le 1 \quad \text{and} \quad |v_i|\le 1 \quad \text{for all } i\in\mathbb{N}. \tag{2.8}$$

Equating the coefficients in (2.6) and (2.7), we deduce that

$$\frac{\theta\big(\gamma+\phi(\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}a_2=\mathcal{G}_1^{\delta}(t)u_1, \tag{2.9}$$

$$\frac{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}a_3$$

$$+\frac{\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\left[\gamma(\gamma-1)+\phi(\mu+1)\big(2\gamma+(\phi-1)(\mu+1)\big)-2\big(\gamma+\phi(3\mu+1)\big)\right]}{2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}a_2^2$$

$$=\mathcal{G}_1^{\delta}(t)u_2+\mathcal{G}_2^{\delta}(t)u_1^2, \tag{2.10}$$

$$-\frac{\theta\big(\gamma+\phi(\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}a_2=\mathcal{G}_1^{\delta}(t)v_1 \tag{2.11}$$

and

$$\frac{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}(2a_2^2-a_3)$$

$$+\frac{\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\left[\gamma(\gamma-1)+\phi(\mu+1)\big(2\gamma+(\phi-1)(\mu+1)\big)-2\big(\gamma+\phi(3\mu+1)\big)\right]}{2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}a_2^2$$

$$=\mathcal{G}_1^{\delta}(t)v_2+\mathcal{G}_2^{\delta}(t)v_1^2. \tag{2.12}$$

From (2.9) and (2.11), we conclude that

$$u_1=-v_1 \tag{2.13}$$

and

$$\frac{2\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}a_2^2=\left(\mathcal{G}_1^{\delta}(t)\right)^2(u_1^2+v_1^2). \tag{2.14}$$

Adding (2.10) to (2.12), yields

$$\frac{\theta\Gamma(\eta+\theta)\Gamma(\eta+\lambda)}{\Gamma(\eta)\Gamma(\lambda)}\Omega(\gamma,\phi,\mu)a_2^2 = \mathcal{G}_1^\delta(t)(u_2+v_2)+\mathcal{G}_2^\delta(t)(u_1^2+v_1^2),$$

where $\Omega(\gamma,\phi,\mu,\eta,\theta,\lambda)$ is given by (2.1). Consequently, we have

$$\left[\frac{\theta\Gamma(\eta+\theta)\Gamma(\eta+\lambda)}{\Gamma(\eta)\Gamma(\lambda)}\Omega(\gamma,\phi,\mu)-\frac{2\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\mathcal{G}_2^\delta(t)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^2}\right]a_2^2$$

$$= \mathcal{G}_1^\delta(t)(u_2+v_2). \tag{2.15}$$

Further computations using (1.3), (2.8) and (2.15), we obtain

$$|a_2| \le \frac{2|\delta|t\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\sqrt{2|\delta|t}}{\sqrt{\left|\begin{array}{c}\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -2\left[\begin{array}{c}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\Omega(\gamma,\phi,\mu)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2\end{array}\right|}}.$$

Next, if we subtract (2.12) from (2.10), we deduce that

$$\frac{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}(a_3-a_2^2)$$

$$= \mathcal{G}_1^\delta(t)(u_2-v_2)+\mathcal{G}_2^\delta(t)(u_1^2-v_1^2). \tag{2.16}$$

In view of (2.13) and (2.14), we get from (2.16)

$$a_3 = \frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^2}{2\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}(u_1^2+v_1^2)$$
$$+\frac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)\mathcal{G}_1^\delta(t)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}(u_2-v_2).$$

Thus applying (1.3), we obtain

$$|a_3| \le \frac{4\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\delta^2t^2}{\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}$$
$$+\frac{2\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)|\delta|t}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}.$$

Putting $\phi = 0$ and $\gamma = 1$ in Theorem 2.1, we demonstrate the next outcome:

Corollary 2.1. *For $t \in \left(\frac{1}{2}, 1\right]$ and δ is a nonzero real constant, let $f \in \mathcal{A}$ be in the family $\mathfrak{T}_\Sigma(\eta, \theta, \gimel, t, \delta)$. Then*

$$|a_2| \leq \frac{2|\delta|t\Gamma(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\sqrt{2|\delta|t}}{\sqrt{\left|\begin{matrix} \delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -2\left[\begin{matrix}\delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\mathfrak{S}(\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{matrix}\right]t^2\end{matrix}\right|}}$$

and

$$|a_3| \leq \frac{4\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\delta^2t^2}{\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)} + \frac{2\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)|\delta|t}{\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)},$$

where

$$\mathfrak{S}(\eta,\theta,\gimel) = \frac{2(\theta+1)\Gamma(\gimel+2)}{\Gamma(\eta+\theta+\gimel+2)} + \frac{-2\theta\Gamma(\eta+\theta)\Gamma^2(\gimel+1)\Gamma(\eta+\gimel)}{\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)}.$$

Theorem 2.2. *For $0 \leq \sigma \leq 1$, $t \in \left(\frac{1}{2}, 1\right]$ and δ is a nonzero real constant, let $f \in \mathcal{A}$ be in the family $\mathbb{S}_\Sigma(\sigma, \eta, \theta, \gimel, t, \delta)$. Then*

$$|a_2| \leq \frac{\Gamma(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)|\delta|t\sqrt{|\delta|t}}{\sqrt{\left|\begin{matrix} \delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\left[\begin{matrix}2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{matrix}\right]t^2\end{matrix}\right|}}$$

and

$$|a_3| \leq \frac{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\delta^2t^2}{\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)} + \frac{2\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)|\delta|t}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)},$$

where

$$\Upsilon(\sigma,\eta,\theta,\gimel) = \frac{3(\theta+1)(2\sigma+1)\Gamma(\gimel+2)}{\Gamma(\eta+\theta+\gimel+2)}$$

$$-\frac{4\theta(\sigma+1)^2\Gamma(\eta+\theta)\Gamma^2(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}. \tag{2.17}$$

Proof. Let $f \in \mathbb{S}_\Sigma(\sigma,\eta,\theta,\lambda,t,\delta)$. Then there exist two holomorphic functions $u, v : U \longrightarrow U$

$$\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)' + (2\sigma+1)z^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'' + \sigma z^3\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'''}{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)' + \sigma z^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)''}$$

$$= 1 + \mathcal{G}_1^\delta(t)u(z) + \mathcal{G}_2^\delta(t)u^2(z) + \cdots \tag{2.18}$$

and

$$\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)' + (2\sigma+1)w^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'' + \sigma w^3\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'''}{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)' + \sigma w^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)''}$$

$$= 1 + \mathcal{G}_1^\delta(t)v(w) + \mathcal{G}_2^\delta(t)v^2(w) + \cdots . \tag{2.19}$$

where u and v have the forms (2.2) and (2.3). Combining (2.18) and (2.19), yield

$$\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)' + (2\sigma+1)z^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'' + \sigma z^3\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'''}{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)' + \sigma z^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)''}$$

$$= 1 + \mathcal{G}_1^\delta(t)u_1 z + \left[\mathcal{G}_1^\delta(t)u_2 + \mathcal{G}_2^\delta(t)u_1^2\right]z^2 + \cdots \tag{2.20}$$

and

$$\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)' + (2\sigma+1)w^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'' + \sigma w^3\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'''}{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)' + \sigma w^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)''}$$

$$= 1 + \mathcal{G}_1^\delta(t)v_1 w + \left[\mathcal{G}_1^\delta(t)v_2 + \mathcal{G}_2^\delta(t)v_1^2\right]w^2 + \cdots . \tag{2.21}$$

Equating the coefficients in (2.20) and (2.21), we deduce that

$$\frac{2\theta(\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}a_2 = \mathcal{G}_1^\delta(t)u_1, \tag{2.22}$$

$$\frac{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}a_3$$

$$-\frac{4\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)}{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)}a_2^2=\mathcal{G}_1^\delta(t)u_2+\mathcal{G}_2^\delta(t)u_1^2, \tag{2.23}$$

$$-\frac{2\theta(\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+1)\Gamma(\eta+\gimel)}{\Gamma(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)}a_2=\mathcal{G}_1^\delta(t)v_1 \tag{2.24}$$

and

$$\frac{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}(2a_2^2-a_3)$$

$$-\frac{4\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)}{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)}a_2^2=\mathcal{G}_1^\delta(t)v_2+\mathcal{G}_2^\delta(t)v_1^2. \tag{2.25}$$

In view of (2.22) and (2.24), we have

$$u_1=-v_1 \tag{2.26}$$

and

$$\frac{8\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)}{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)}a_2^2=\left(\mathcal{G}_1^\delta(t)\right)^2(u_1^2+v_1^2). \tag{2.27}$$

If we add (2.23) to (2.25), we conclude that

$$\frac{2\theta\Gamma(\eta+\theta)\Gamma(\eta+\gimel)}{\Gamma(\eta)\Gamma(\gimel)}\Upsilon(\sigma,\eta,\theta,\gimel)a_2^2=\mathcal{G}_1^\delta(t)(u_2+v_2)+\mathcal{G}_2^\delta(t)(u_1^2+v_1^2), \tag{2.28}$$

where $\Upsilon(\sigma,\eta,\theta,\gimel)$ is given by (2.17).

By substitute the value of $u_1^2+v_1^2$ from (2.27) in (2.28), yields

$$\left[\frac{2\theta\Gamma(\eta+\theta)\Gamma(\eta+\gimel)}{\Gamma(\eta)\Gamma(\gimel)}\Upsilon(\sigma,\eta,\theta,\gimel)-\frac{8\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\mathcal{G}_2^\delta(t)}{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\left(\mathcal{G}_1^\delta(t)\right)^2}\right]a_2^2$$
$$=\mathcal{G}_1^\delta(t)(u_2+v_2),$$

or equivalently

$$a_2^2=\frac{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\left(\mathcal{G}_1^\delta(t)\right)^3(u_2+v_2)}{\begin{array}{c}2\theta\Upsilon(\sigma,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\left(\mathcal{G}_1^\delta(t)\right)^2\\-8\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\mathcal{G}_2^\delta(t)\end{array}}, \tag{2.29}$$

Further computations using (1.3), (2.7) and (2.29), we obtain

$$|a_2| \leq \frac{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)|\delta|t\sqrt{|\delta|t}}{\sqrt{\left|\begin{array}{c}\delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)-\\ \left[\begin{array}{c}2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2\end{array}\right|}}.$$

Next, if we subtract (2.25) from (2.23), we deduce that

$$\begin{aligned}&\frac{6\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}(a_3-a_2^2)\\ &= \mathcal{G}_1^\delta(t)(u_2-v_2)+\mathcal{G}_2^\delta(t)(u_1^2-v_1^2).\end{aligned} \tag{2.30}$$

In view of (2.26) and (2.27), we get from (2.30)

$$\begin{aligned}a_3 &= \frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^2}{8\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}(u_1^2+v_1^2)\\ &\quad+\frac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)\mathcal{G}_1^\delta(t)}{6\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}(u_2-v_2).\end{aligned}$$

Thus applying (1.3), we obtain

$$\begin{aligned}|a_3| &\leq \frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\delta^2t^2}{\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}\\ &\quad+\frac{2\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)|\delta|t}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}.\end{aligned}$$

Putting $\sigma = 0$ in Theorem 2.2, we demonstrate the next outcome:

Corollary 2.2. *For $t \in \left(\frac{1}{2}, 1\right]$ and δ is a nonzero real constant, let $f \in \mathcal{A}$ be in the family $\mathfrak{H}_\Sigma(\eta,\theta,\lambda,t,\delta)$. Then*

$$|a_2| \leq \frac{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)|\delta|t\sqrt{|\delta|t}}{\sqrt{\left|\begin{array}{c}\delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\left[\begin{array}{c}2\delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\mathfrak{A}(\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2\end{array}\right|}}$$

and

$$|a_3| \leq \frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\delta^2t^2}{\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}+\frac{2\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)|\delta|t}{3\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)},$$

where

$$\mathfrak{A}(\eta,\theta,\gimel)=\frac{3(\theta+1)\Gamma(\gimel+2)}{\Gamma(\eta+\theta+\gimel+2)}-\frac{4\theta\Gamma(\eta+\theta)\Gamma^2(\gimel+1)\Gamma(\eta+\gimel)}{\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)}.$$

Next theorems, show “Fekete-Szegö problem” of the families $\mathbb{T}_\Sigma(\gamma,\phi,\mu,,\eta,\theta,\gimel,t,\delta)$ and $\mathbb{S}_\Sigma(\sigma,\eta,\theta,\gimel,t,\delta)$.

Theorem 2.3. *For* $0\le\gamma\le 1, 0\le\phi\le 1, 0\le\mu\le 1,\ t\in\left(\frac{1}{2},1\right],\ \xi\in\mathbb{R}$ *and* δ *is a nonzero real constant, let* $f\in\mathcal{A}$ *be in the family* $\mathbb{T}_\Sigma(\gamma,\phi,\mu,,\eta,\theta,\gimel,t,\delta)$. *Then*

$$|a_3-\xi a_2^2|$$

$$\le\begin{cases}\dfrac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}; \\ \qquad for\ \ |\xi-1|\le\dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}\times \\ \qquad\times\left|\dfrac{\begin{array}{c}\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\\ -2\left[\begin{array}{c}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{array}\right]t^2\end{array}}{2\delta^2t^2\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)}\right| \\ \\ \dfrac{8t^3|\delta^3|\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)|\xi-1|}{\left|\begin{array}{c}\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\\ -2\left[\begin{array}{c}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{array}\right]t^2\end{array}\right|}; \\ \qquad for\ \ |\xi-1|\ge\dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}\times \\ \qquad\times\left|\dfrac{\begin{array}{c}\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\\ -2\left[\begin{array}{c}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{array}\right]t^2\end{array}}{2\delta^2t^2\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)}\right|\end{cases}.$$

Proof. In the light of (2.15) and (2.16), we deduce that

$$a_3 - \xi a_2^2 = (1-\xi)a_2^2 + \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)\mathcal{G}_1^\delta(t)(u_2-v_2)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}$$

$$= (1-\xi)\frac{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\left(\mathcal{G}_1^\delta(t)\right)^3(u_2+v_2)}{\theta\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma(\eta)\Gamma(\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\left(\mathcal{G}_1^\delta(t)\right)^2 - 2\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\mathcal{G}_2^\delta(t)}$$

$$+ \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)\mathcal{G}_1^\delta(t)(u_2-v_2)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}$$

$$= \mathcal{G}_1^\delta(t)\left[\left(\psi(\xi) + \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}\right)u_2 + \left(\psi(\xi) - \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}\right)v_2\right],$$

where

$$\psi(\xi) = \frac{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\left(\mathcal{G}_1^\delta(t)\right)^2(1-\xi)}{\theta\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma(\eta)\Gamma(\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\left(\mathcal{G}_1^\delta(t)\right)^2 - 2\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\mathcal{G}_2^\delta(t)}.$$

According to (1.3), we deduce that

$$\left|a_3 - \xi a_2^2\right| \le \begin{cases} \dfrac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}, \\ 0 \le |\psi(\xi)| \le \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \\ 4t|\delta||\psi(\xi)|, \\ |\psi(\xi)| \ge \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \end{cases}.$$

After some computations, we obtain

$$
|a_3-\xi a_2^2| \le \begin{cases}
\dfrac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}; \\
\quad \text{for } |\xi-1| \le \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \times \\
\quad \times \left| \dfrac{\begin{array}{c}\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -2\begin{bmatrix}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{bmatrix} t^2\end{array}}{2\delta^2 t^2\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)} \right| \\[2ex]
\dfrac{8t^3|\delta^3|\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)|\xi-1|}{\left|\begin{array}{c}\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -2\begin{bmatrix}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{bmatrix} t^2\end{array}\right|}; \\
\quad \text{for } |\xi-1| \ge \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \times \\
\quad \times \left| \dfrac{\begin{array}{c}\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -2\begin{bmatrix}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{bmatrix} t^2\end{array}}{2\delta^2 t^2\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)} \right|
\end{cases} .
$$

Putting $\xi = 1$ in Theorem 2.3, we demonstrate the next outcome:

Corollary 2.3. *For* $0 \le \gamma \le 1, 0 \le \phi \le 1, 0 \le \mu \le 1,\ t \in \left(\frac{1}{2}, 1\right]$ *and* δ *is a nonzero real constant, let* $f \in \mathcal{A}$ *be in the family* $\mathbb{T}_\Sigma(\gamma, \phi, \mu, , \eta, \theta, \gimel, t, \delta)$. *Then*

$$
\left|a_3 - a_2^2\right| \le \frac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}.
$$

Putting $\phi = 0$ and $\gamma = 1$ in Theorem 2.3, we demonstrate the next outcome:

Corollary 2.4. *For* $t \in \left(\frac{1}{2}, 1\right]$, $\xi \in \mathbb{R}$ *and* δ *is a nonzero real constant, let* $f \in \mathcal{A}$ *be in the family* $\mathfrak{T}_\Sigma(\eta, \theta, \beth, t, \delta)$. *Then*

$$\left|a_3 - \xi a_2^2\right| \le \begin{cases} \dfrac{2t|\delta|\Gamma(\eta+\theta+\beth+2)\Gamma(\eta)\Gamma(\beth)}{\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\beth+2)\Gamma(\eta+\beth)}; \\ \qquad \text{for } |\xi - 1| \le \dfrac{\Gamma(\eta+\theta+\beth+2)\Gamma(\eta)\Gamma(\beth)}{2\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\beth+2)\Gamma(\eta+\beth)} \times \\ \qquad \times \left| \dfrac{\delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\beth+1)\Gamma^2(\eta+\beth) - 2\begin{bmatrix} \delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\beth+1)\Gamma^2(\eta+\beth) \\ -\theta\delta^2\mathfrak{S}(\eta,\theta,\beth)\Gamma(\eta+\theta)\Gamma(\eta+\beth)\Gamma^2(\eta+\theta+\beth+1)\Gamma(\eta)\Gamma(\beth) \end{bmatrix} t^2}{2\delta^2 t^2 \Gamma^2(\eta+\theta+\beth+1)\Gamma^2(\eta)\Gamma^2(\beth)} \right| \\ \\ \dfrac{8t^3|\delta^3|\Gamma^2(\eta+\theta+\beth+1)\Gamma^2(\eta)\Gamma^2(\beth)|\xi-1|}{\left| \delta\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\beth+1)\Gamma^2(\eta+\beth) - 2\begin{bmatrix} \delta(\delta+1)\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\beth+1)\Gamma^2(\eta+\beth) \\ -\theta\delta^2\mathfrak{S}(\eta,\theta,\beth)\Gamma(\eta+\theta)\Gamma(\eta+\beth)\Gamma^2(\eta+\theta+\beth+1)\Gamma(\eta)\Gamma(\beth) \end{bmatrix} t^2 \right|}; \\ \quad \text{for } |\xi - 1| \ge \dfrac{\Gamma(\eta+\theta+\beth+2)\Gamma(\eta)\Gamma(\beth)}{2\theta(\theta+1)\left(\gamma+\phi(2\mu+1)\right)\Gamma(\eta+\theta)\Gamma(\beth+2)\Gamma(\eta+\beth)} \times \\ \qquad \times \left| \dfrac{\delta\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\beth+1)\Gamma^2(\eta+\beth) - 2\begin{bmatrix} \delta(\delta+1)\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\beth+1)\Gamma^2(\eta+\beth) \\ -\theta\delta^2\mathfrak{S}(\eta,\theta,\beth)\Gamma(\eta+\theta)\Gamma(\eta+\beth)\Gamma^2(\eta+\theta+\beth+1)\Gamma(\eta)\Gamma(\beth) \end{bmatrix} t^2}{2\delta^2 t^2 \Gamma^2(\eta+\theta+\beth+1)\Gamma^2(\eta)\Gamma^2(\beth)} \right| \end{cases}.$$

Theorem 2.4. *For* $0 \le \sigma \le 1$, $t \in \left(\frac{1}{2}, 1\right]$, $\xi \in \mathbb{R}$ *and* δ *is a nonzero real constant, let* $f \in \mathcal{A}$ *be in the family* $\mathbb{S}_\Sigma(\sigma, \eta, \theta, \beth, t, \delta)$. *Then*

$$\left|a_3 - \xi a_2^2\right|$$

$$\leq \begin{cases} \dfrac{2t|\delta|\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}; \\ \qquad \text{for } |\xi-1| \leq \dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)} \times \\ \qquad \times \left| \dfrac{\delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) - \begin{bmatrix} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda) \end{bmatrix} t^2}{\delta^2 t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)} \right| \\ \\ \dfrac{2t^3|\delta^3|\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)|\xi-1|}{\left| \delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) - \begin{bmatrix} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda) \end{bmatrix} t^2 \right|} \\ \qquad \text{for } |\xi-1| \geq \dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)} \times \\ \qquad \times \left| \dfrac{\delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) - \begin{bmatrix} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda) \end{bmatrix} t^2}{\delta^2 t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)} \right| \end{cases}.$$

Proof. In view of (2.29) and (2.30), we deduce that

$$a_3 - \xi a_2^2 = (1-\xi)a_2^2 + \frac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)\mathcal{G}_1^\delta(t)(u_2-v_2)}{6\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}$$

$$= (1-\xi)\frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^3(u_2+v_2)}{\begin{matrix} 2\theta\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^2 \\ -8\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\mathcal{G}_2^\delta(t) \end{matrix}}$$

$$+\frac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)\mathcal{G}_1^\delta(t)(u_2-v_2)}{6\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}$$

$$= \frac{\mathcal{G}_1^{\delta}(t)}{2}\left[\left(\varphi(\xi) + \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}\right)u_2 \right.$$
$$\left. + \left(\varphi(\xi) - \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}\right)v_2\right],$$

where

$$\varphi(\xi) = \frac{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\left(\mathcal{G}_1^{\delta}(t)\right)^2(1-\xi)}{\begin{array}{c}\theta\Upsilon(\sigma,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\left(\mathcal{G}_1^{\delta}(t)\right)^2 \\ -4\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\mathcal{G}_2^{\delta}(t)\end{array}}.$$

According to (1.3), we deduce that

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \dfrac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}, \\ \quad 0 \leq |\varphi(\xi)| \leq \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \\ \\ 2t|\delta||\varphi(\xi)|, \\ \quad |\varphi(\xi)| \geq \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \end{cases}.$$

After some computations, we obtain

$$\left|a_3-\xi a_2^2\right| \leq \begin{cases} \dfrac{2t|\delta|\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}; \\ \qquad \text{for } |\xi-1| \leq \dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)} \times \\ \qquad \times \left| \dfrac{\delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) - \left[\begin{array}{c} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda) \end{array}\right] t^2}{\delta^2 t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)} \right| \\ \dfrac{2t^3|\delta^3|\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)|\xi-1|}{\left| \delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) - \left[\begin{array}{c} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda) \end{array}\right] t^2 \right|} \\ \qquad \text{for } |\xi-1| \geq \dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)} \times \\ \qquad \times \left| \dfrac{\delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) - \left[\begin{array}{c} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda) \end{array}\right] t^2}{\delta^2 t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)} \right| \end{cases}.$$

Putting $\xi = 1$ in Theorem 2.4, we demonstrate the next outcome:

Corollary 2.5. *For* $0 \leq \sigma \leq 1$, $t \in \left(\frac{1}{2}, 1\right]$ *and* δ *is a nonzero real constant, let* $f \in \mathcal{A}$ *be in the family* $\mathbb{S}_\Sigma(\sigma,\eta,\theta,\lambda,t,\delta)$. *Then*

$$\left|a_3-a_2^2\right| \leq \frac{2t|\delta|\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}.$$

Putting $\sigma = 0$ in Theorem 2.4, we demonstrate the next outcome:

Corollary 2.6. *For* $t \in \left(\frac{1}{2}, 1\right]$, $\xi \in \mathbb{R}$ *and* δ *is a nonzero real constant, let* $f \in \mathcal{A}$ *be in the family* $\mathfrak{H}_\Sigma(\eta,\theta,\lambda,t,\delta)$. *Then*

$$\left|a_3-\xi a_2^2\right|$$
$$\leq\begin{cases}\dfrac{2t|\delta|\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}; \\ \qquad \text{for } |\xi-1|\leq\dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}\times \\ \qquad \times\left|\dfrac{\begin{array}{c}\delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\left[\begin{array}{c}2\delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\mathfrak{A}(\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2\end{array}}{\delta^2t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}\right| \\ \dfrac{2t^3|\delta^3|\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)|\xi-1|}{\left|\begin{array}{c}\delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\left[\begin{array}{c}2\delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\mathfrak{A}(\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2\end{array}\right|} \\ \qquad \text{for } |\xi-1|\geq\dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}\times \\ \qquad \times\left|\dfrac{\begin{array}{c}\delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\left[\begin{array}{c}2\delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\mathfrak{A}(\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2\end{array}}{\delta^2t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}\right|\end{cases}.$$

Conclusion

The primary objective was to create the families $\mathbb{T}_\Sigma(\gamma,\phi,\mu,,\eta,\theta,\lambda,t,\delta)$ and $\mathbb{S}_\Sigma(\sigma,\eta,\theta,\lambda,t,\delta)$ of bi-univalent functions which governed by Gegenbauer polynomials. We generated Taylor coefficient inequalities for functions in the families $\mathbb{T}_\Sigma(\gamma,\phi,\mu,,\eta,\theta,\lambda,t,\delta)$ and $\mathbb{S}_\Sigma(\sigma,\eta,\theta,\lambda,t,\delta)$ and viewed the famous Fekete-Szegö problem.

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Burhan Distribution with Structural Properties and Applications in Distinct Areas of Science

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Abstract

In this work a novel distribution has been explored referred as Burhan distribution. This distribution is obtained through convex combination of exponential and gamma distribution to analyse complex real-life data. The distinct structural properties of the formulated distribution have been derived and discussed. The behaviour of probability density function (pdf) and cumulative distribution function (cdf) are illustrated through different graphs. The estimation of the established distribution parameters are performed by maximum likelihood estimation method. Eventually the versatility of the established distribution is examined through two real life data sets.

1. Introduction

In biomedicine, engineering, economics, and other fields of science, statistical distribution plays essential role in modelling data. The exponential and gamma distributions are popular distributions for assessing statistical data and are regarded as life time distributions. Among these distributions, the exponential distribution has one parameter and several attractive statistical characteristics, such as memory less and a constant hazard rate. Various extensions of these distributions have been made in the statistical literature to lead greater flexibility. Lindley formulated a one parameter life time distribution with the following probability density function in (1958).

$$f(y,\theta)=\frac{\theta^2}{\theta+1}(1+y)e^{-\theta y};\quad y>0,\theta>0$$

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In recent years, researchers have worked extensively upon Lindley distribution, formulating one and two parameter distributions. For the modelling of diverse complex data, Ghitney et al. [2] conducted an in-depth study on the Lindley distribution, demonstrating that the Lindley distribution outranks the exponential distribution in terms of waiting times for bank customers. They also demonstrate that the contours of the hazard rate function of a Lindley distribution are increasing whereas the mean residual life is decreasing. Zakerzadeh and Dolati [3], Nadarajah et al. [4], they extended Lindley distribution with addition of new parameters and expounded the performance of the extended distribution through data sets. In recent years many authors have made different contributions to modify the Lindley distribution. Merovci [5], has introduced transmuted Lindley distribution and discussed its several properties. Sharma et al has introduced the inverse of the Lindley distribution and propounded its different properties. Shanker et al [6] developed a new life time distribution named Akash distribution which were proved superior then exponential and Lindley distribution. Sen et al. [7] has proposed a one parameter distribution called xgamma distribution and they have proved by an application that xgamma distribution provide better fit than exponential distribution. K.K Shukla [8], have suggested the Pranav distribution and studied its different properties. Aijaz et al. [9] has introduced transmuted inverse Lindley distributions and estimates its properties. Aijaz et al. [10] has developed a new distribution and name it Hamza distribution and studied its different properties. When it comes to analysing more complex data, these distributions have both advantages and limitations. In this study, an attempt is made to establish a new three parameter distribution that is considerably more pliable and produces better results than previous ones. The newly defined three parameter distribution's probability density function is as follows.

$$f(y,\alpha,\beta,\theta)=\frac{\theta^{\beta+1}}{\alpha\theta^{\beta}+\Gamma(\beta)}\left(\alpha+\frac{y^{\beta}}{\beta}\right)e^{-\theta y};\;\; y>0,\alpha,\beta,\theta>0. \tag{1.1}$$

The suggested distribution is referred to as the Burhan distribution, and it has been developed by combining two distributions, exponential (θ) and gamma (β,θ), by employing the linear combination method.

$$f(y,\alpha,\beta,\theta)=\rho g_1(y,\theta)+(1-\rho)g_2(y,\beta,\theta),$$

Where $$\rho=\frac{\theta^{\beta+1}}{\alpha\theta^{\beta}+\Gamma(\beta)}$$

$$g_1(y,\theta)=\theta e^{-\theta y};\; y>0,\theta>0;\quad g_2(y,\theta)=\frac{y^{\beta}}{\Gamma(\beta+1)}e^{-\theta y};\; y>0,\beta,\theta>0.$$

Figures (1.1) and (1.2) expound few layouts of pdf of Burhan distribution for varying parameters.

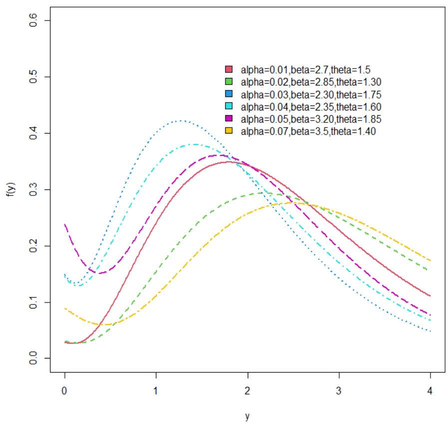

Figure 1.1: pdf of Burhan distribution under different values to parameters.

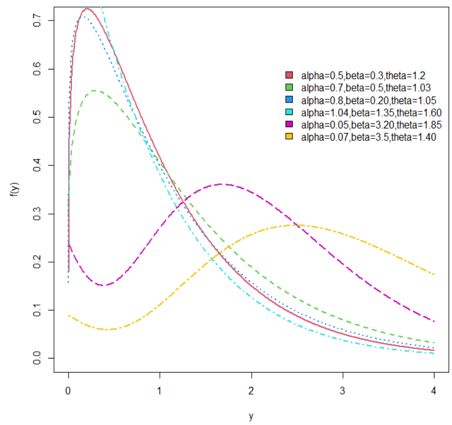

Figure 1.2: pdf of Burhan distribution under different values to parameters.

The associated cdf of equation (1.1) is given by

$$F(y,\alpha,\beta,\theta) = 1 - \frac{\left(\alpha\beta\theta^{\beta}e^{-\theta y} + \Gamma(\beta+1,\theta y)\right)}{\beta\left(\alpha\theta^{\beta} + \Gamma(\beta)\right)}; \quad y > 0, \beta, \theta > 0. \tag{1.2}$$

Figures (1.3) and (1.4) expounds few layouts of cdf of Burhan distribution for varying parameters

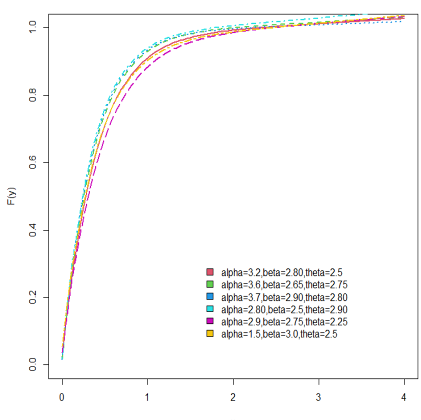

Figure 1.3: cdf of Burhan distribution under different values to parameters.

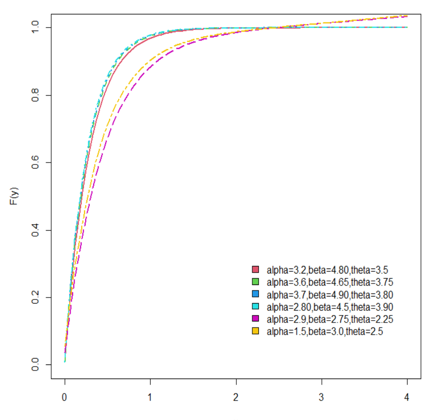

Figure 1.4: cdf of Burhan distribution under different values to parameters.

2. Mathematical Properties of Burhan Distribution

2.1. Moments of Burhan distribution

Let Y be a random variable follows Burhan distribution. Then r^{th} moment denoted by μ_r' is given as

$$\mu_r' = E(Y^r) = \int_0^\infty y^r f(y, \alpha, \beta, \theta)\, dy$$

$$= \int_0^\infty y^r f(y, \alpha, \beta, \theta)\, dy$$

$$= \int_0^\infty y^r \left(\alpha + \frac{y^\beta}{\beta}\right) e^{-\theta y} dy$$

$$= \alpha \int_0^\infty y^r\, e^{-\theta y} + \frac{1}{\beta} \int_0^\infty y^{r+\beta}\, e^{-\theta y} dy.$$

After solving the integral, we have

$$\mu_r' = \frac{\alpha\beta\theta^\beta \Gamma(r+1) + \Gamma(r+\beta+1)}{\beta\theta^{r+\beta+1}}.$$

Substituting $r = 1,2,3,4$ we obtain first four moments about origin as

$$\mu_1' = \frac{\alpha\theta^\beta + (\beta+1)\Gamma(\beta)}{\theta^{\beta+2}}$$

$$\mu_2' = \frac{2\alpha\theta^\beta + (\beta+2)(\beta+1)\Gamma(\beta)}{\theta^{\beta+3}}$$

$$\mu_3' = \frac{6\alpha\theta^\beta + (\beta+3)(\beta+2)(\beta+1)\Gamma(\beta)}{\theta^{\beta+4}}$$

$$\mu_4' = \frac{24\alpha\theta^\beta + (\beta+4)(\beta+3)(\beta+2)(\beta+1)\Gamma(\beta)}{\theta^{\beta+5}}.$$

2.2. Moment generating function of Burhan distribution

Let Y be a random variable follows Burhan distribution. Then the moment generating function of the distribution denoted by $M_Y(t)$ is given

$$M_Y(t) = E(e^{ty}) = \int_0^\infty e^{ty} f(y, \alpha, \beta, \theta)\, dy.$$

Using Taylor's series

$$= \int_0^\infty \left(1 + ty + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \ldots\right) f(y, \alpha, \beta, \theta) dy$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty y^r f(y, \alpha, \beta, \theta)\, dy$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} E(Y^r)$$

$$M_Y(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\alpha\beta\theta^\beta \Gamma(r+1) + \Gamma(r+\beta+1)}{\beta\theta^{r+\beta+1}}.$$

3. Renyi Entropy of Burhan Distribution

If Y denotes a continuous random variable having probability density function $f(y, \alpha, \beta, \theta)$. Then Renyi entropy is defined as

$$T_R(\gamma) = \frac{1}{1-\gamma} log\left\{\int_0^\infty f^\gamma(y) dy\right\},$$

where $\gamma > 0$ and $\gamma \neq 1$.

Thus, the Renyi entropy of Burhan distribution (1.1) is given as

$$T_R(\gamma) = \frac{1}{1-\gamma} log\left\{\frac{\theta^{(\beta+1)\gamma}}{\left(\alpha\theta^\beta + \Gamma(\beta)\right)^\gamma} \int_0^\infty \left(\alpha + \frac{y^\beta}{\beta}\right)^\gamma e^{-\gamma\theta y} dy\right\}$$

$$= \frac{1}{1-\gamma} log\left\{\frac{\theta^{(\beta+1)\gamma}\alpha^\gamma}{\left(\alpha\theta^\beta + \Gamma(\beta)\right)^\gamma} \int_0^\infty \left(1 + \frac{y^\beta}{\beta}\right)^\gamma e^{-\gamma\theta y} dy\right\}.$$

Using generalized binomial theorem, we have

$$T_R(\gamma) = \frac{1}{1-\gamma} log\left\{\sum_{r=0}^{\infty} \binom{\gamma}{r} \left(\frac{\alpha\theta^{(\beta+1)}}{\alpha\theta^\beta + \Gamma(\beta)}\right)^\gamma \frac{1}{(\alpha\beta)^r} \int_0^\infty y^{\beta r} e^{-\gamma\theta y} dy\right\}.$$

After solving the integral, we get

$$T_R(\gamma) = \frac{1}{1-\gamma} log\left\{\sum_{r=0}^{\infty} \binom{\gamma}{r} \left(\frac{\alpha\theta^{(\beta+1)}}{\alpha\theta^\beta + \Gamma(\beta)}\right)^\gamma \frac{\Gamma(\beta r + 1)}{(\alpha\beta)^r (\gamma\theta)^{\beta r+1}}\right\}.$$

4. Tsallis Entropy of Burhan Distribution

Tsallis entropy of order γ for Burhan distribution (1.1) is given as

$$S_\gamma = \frac{1}{\gamma - 1}\left\{1 - \int_0^\infty f^\gamma(y)dy\right\},$$

where $\gamma > 0$ and $\gamma \neq 1$

$$S_\gamma = \frac{1}{\gamma - 1}\left\{1 - \frac{\theta^{(\beta+1)\gamma}}{\left(\alpha\theta^\beta + \Gamma(\beta)\right)^\gamma}\int_0^\infty \left(\alpha + \frac{y^\beta}{\beta}\right)^\gamma e^{-\gamma\theta y}dy\right\}.$$

After solving the integral, we get

$$S_\gamma = \frac{1}{\gamma - 1}\left(1 - \sum_{r=0}^{\infty}\binom{\gamma}{r}\left(\frac{\alpha\theta^{(\beta+1)}}{\alpha\theta^\beta + \Gamma(\beta)}\right)^\gamma \frac{\Gamma(\beta r + 1)}{(\alpha\beta)^r(\gamma\theta)^{\beta r+1}}\right).$$

5. Mean Deviation from Mean of Burhan Distribution

The quantity of scattering in a population is evidently measured to some extent by the totality of the deviations. Let Y be a random variable from Burhan distribution with mean μ. Then the mean deviation from mean is defined as

$$\begin{aligned}
D(\mu) &= E(|Y - \mu|) \\
&= \int_0^\infty |Y - \mu|\, f(y)dy \\
&= \int_0^\mu (\mu - y)\, f(y)dy + \int_\mu^\infty (y - \mu)\, f(y)dy \\
&= \mu\int_0^\mu f(y)\, dy - \int_0^\mu yf(y)\, dy + \int_\mu^\infty yf(y)\, dy - \int_\mu^\infty \mu f(y)\, dy \\
&= \mu F(\mu) - \int_0^\mu yf(y)\, dy - \mu[1 - F(\mu)] + \int_\mu^\infty yf(y)\, dy \\
&= 2\mu F(\mu) - 2\int_0^\mu yf(y)\, dy. \qquad (5.1)
\end{aligned}$$

Now

$$\int_0^\mu yf(y)\, dy = \int_0^\mu y\frac{\theta^{\beta+1}}{\alpha\theta^\beta + \Gamma(\beta)}\left(\alpha + \frac{y^\beta}{\beta}\right)e^{-\theta y}\, dy$$

$$= \frac{\theta^{\beta+1}}{\alpha\theta^{\beta} + \Gamma(\beta)} \left\{ \alpha \int_0^{\mu} y e^{-\theta y} dy + \frac{1}{\beta} \int_0^{\mu} y^{\beta+1} e^{-\theta y} dy \right\}.$$

After solving the integral, we get

$$\int_0^{\mu} y f(y)\, dy = \frac{\theta^{\beta+1}}{\alpha\theta^{\beta} + \Gamma(\beta)} \left\{ \frac{\alpha}{\theta^2} \gamma(2, \theta\mu) + \frac{1}{\beta\theta^{\beta+2}} \gamma(\beta+2, \mu\theta) \right\}. \quad (5.2)$$

Using equation (5.2) and (1.2) in equation (5.1), we obtain

$$D(\mu) = 2\mu - \frac{2\mu\left(\alpha\beta\theta^{\beta} e^{-\theta\mu} + \Gamma(\beta+1, \theta\mu)\right)}{\beta\left(\alpha\theta^{\beta} + \Gamma(\beta)\right)}$$

$$- \frac{2\theta^{\beta+1}}{\alpha\theta^{\beta} + \Gamma(\beta)} \left\{ \frac{\alpha}{\theta^2} \gamma(2, \theta\mu) + \frac{1}{\beta\theta^{\beta+2}} \gamma(\beta+2, \mu\theta) \right\}.$$

6. Mean Deviation from Median of Burhan Distribution

Let Y be a random variable from Burhan distribution with median M. Then the mean deviation from median is defined as

$$D(M) = E(|Y - M|)$$

$$= \int_0^{\infty} |Y - M| f(y) dy$$

$$= \int_0^{M} (M - y) f(y) dy + \int_M^{\infty} (y - M) f(y) dy$$

$$= MF(M) - \int_0^{M} y f(y)\, dy - M[1 - F(M)] + \int_M^{\infty} y f(y)\, dy$$

$$= \mu - 2 \int_0^{M} y f(y)\, dy. \quad (6.1)$$

Now

$$\int_0^{M} y f(y)\, dy = \int_0^{M} y \frac{\theta^{\beta+1}}{\alpha\theta^{\beta} + \Gamma(\beta)} \left(\alpha + \frac{y^{\beta}}{\beta} \right) e^{-\theta y}\, dy.$$

After solving the integral, we get

$$\int_0^M y f(y)\, dy = \frac{\theta^{\beta+1}}{\alpha\theta^\beta + \Gamma(\beta)}\left\{\frac{\alpha}{\theta^2}\gamma(2, \theta M) + \frac{1}{\beta\theta^{\beta+2}}\gamma(\beta+2, M\theta)\right\}. \tag{6.2}$$

Using equation (6.2) and (1.2) in equation (6.1), we obtain

$$D(M) = \mu - \frac{2\theta^{\beta+1}}{\alpha\theta^\beta + \Gamma(\beta)}\left\{\frac{\alpha}{\theta^2}\gamma(2, \theta M) + \frac{1}{\beta\theta^{\beta+2}}\gamma(\beta+2, M\theta)\right\}.$$

7. Reliability Measures

Suppose Y be a continuous random variable with cdf $F(y)$, $y \geq 0$. Then its reliability function which is also called survival function is defined as

$$S(y) = p_r(Y > y) = \int_y^\infty f(y)\, dy = 1 - F(y).$$

Therefore, the survival function for Burhan distribution is given as

$$S(y, \alpha, \beta, \theta) = 1 - F(y, \alpha, \beta, \theta)$$

$$= \frac{\left(\alpha\beta\theta^\beta e^{-\theta y} + \Gamma(\beta+1, \theta y)\right)}{\beta\left(\alpha\theta^\beta + \Gamma(\beta)\right)}. \tag{7.1}$$

The hazard rate function of a random variable y is given as

$$h(y, \alpha, \beta, \theta) = \frac{f(y, \alpha, \beta, \theta)}{S(y, \alpha, \beta, \theta)}. \tag{7.2}$$

Using equation (1.1) and equation (7.1) in (7.2), we get

$$h(y, \alpha, \beta, \theta) = \frac{\beta\theta^{\beta+1}\left(\alpha + \frac{y^\beta}{\beta}\right)e^{-\theta y}}{\left(\alpha\beta\theta^\beta e^{-\theta y} + \Gamma(\beta+1, \theta y)\right)}.$$

Figure (7.1) and (7.2) expound few layouts of hazard rate function of Burhan distribution for varying parameters.

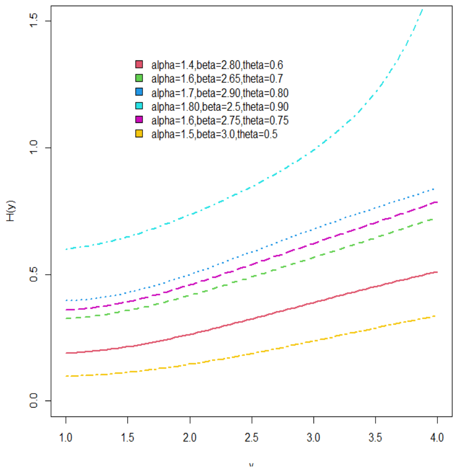

Figure 7.1: HRF of Burhan distribution under different values to parameters.

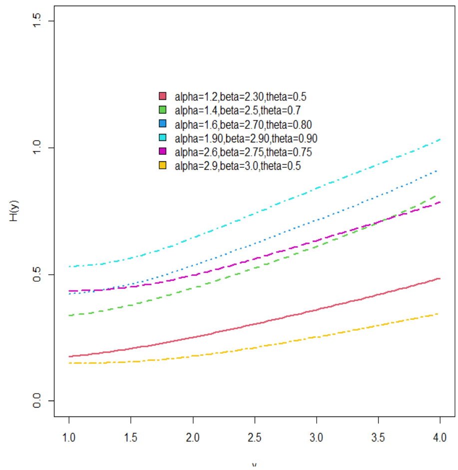

Figure 7.2: HRF of Burhan distribution under different values to parameters.

The cumulative hazard rate function is given as

$$H(y,\alpha,\beta,\theta) = -\log[F(y,\alpha,\beta,\theta)].$$

Using equation (1.2), we get

$$H(y,\alpha,\beta,\theta) = -\log\left[1 - \frac{\left(\alpha\beta\theta^{\beta}e^{-\theta y} + \Gamma(\beta+1,\theta y)\right)}{\beta\left(\alpha\theta^{\beta} + \Gamma(\beta)\right)}\right].$$

Mean residual function of random variable y can be obtained as

$$m(y,\alpha,\beta,\theta) = \frac{1}{S(y,\alpha,\beta,\theta)}\int_y^{\infty} t f(t,\alpha,\beta,\theta)\,dt - y$$

$$= \frac{\beta\left(\alpha\theta^{\beta} + \Gamma(\beta)\right)}{\alpha\beta\theta^{\beta}e^{-\theta y} + \Gamma(\beta+1,\theta y)}\int_0^{\infty}\frac{\theta^{\beta+1}}{\alpha\theta^{\beta}+\Gamma(\beta)}\,t\left(\alpha + \frac{t^{\beta}}{\beta}\right)e^{-\theta t}dt - y$$

$$= \frac{\beta\theta^{\beta+1}}{\alpha\beta\theta^{\beta}e^{-\theta y} + \Gamma(\beta+1,\theta y)}\left\{\int_0^{\infty} te^{-\theta t}\,dt + \frac{1}{\beta}\int_0^{\infty} t^{\beta+1}e^{-\theta t}dt\right\} - y.$$

After solving the integral, we get

$$m(y,\alpha,\beta,\theta) = \frac{\alpha\beta\theta^{\beta}(\theta y+1)e^{-\theta y} + \Gamma(\beta+2,\theta y)}{\theta\left(\alpha\beta\theta^{\beta}e^{-\theta y} + \Gamma(\beta+1,\theta y)\right)} - y.$$

8. Order Statistics of Burhan Distribution

Let $Y_{(1)}, Y_{(2)}, Y_{(3)}, \ldots, Y_{(n)}$ denotes the order statistics of a random sample drawn from a continuous distribution with cdf $F(y)$ and pdf $f(y)$, then the pdf of $Y_{(k)}$ is given by

$$f_{Y(k)}(Y,\theta) = \frac{n!}{(k-1)!\,(n-k)!} f_Y(y)[F_Y(y)]^{k-1}[1-F_Y(y)]^{n-k} \quad k = 1,2,3\ldots, n. \tag{8.1}$$

Substitute the equation (1.1) and (2.2) in equation (8.1), we obtain the probability function of k^{th} order statistics of Burhan distribution is given by

$$f_{Y(k)}(y,\theta) = \frac{n!}{(k-1)!\,(n-k)!} \frac{\theta^{\beta+1}}{\alpha\theta^{\beta}+\Gamma(\beta)}\left(\alpha+\frac{y^{\beta}}{\beta}\right)e^{-\theta y}\left[1-\frac{\left(\alpha\beta\theta^{\beta}e^{-\theta y}+\Gamma(\beta+1,\theta y)\right)}{\beta\left(\alpha\theta^{\beta}+\Gamma(\beta)\right)}\right]^{k-1}$$

$$\left[\frac{\left(\alpha\beta\theta^{\beta}e^{-\theta y}+\Gamma(\beta+1,\theta y)\right)}{\beta\left(\alpha\theta^{\beta}+\Gamma(\beta)\right)}\right]^{n-k}. \tag{8.2}$$

Then, the pdf of first order statistics $Y_{(1)}$ of Burhan distribution given by

$$f_{Y(1)}(y,\theta) = \frac{n\theta^{\beta+1}}{\alpha\theta^{\beta}+\Gamma(\beta)}\left(\alpha+\frac{y^{\beta}}{\beta}\right)e^{-\theta y}\left[\frac{\left(\alpha\beta\theta^{\beta}e^{-\theta y}+\Gamma(\beta+1,\theta y)\right)}{\beta\left(\alpha\theta^{\beta}+\Gamma(\beta)\right)}\right]^{1-k}.$$

And the pdf of nth order statistics $Y_{(n)}$ of Burhan distribution is given by

$$f_{Y(n)}(y,\theta) = \frac{n\theta^{\beta+1}}{\alpha\theta^{\beta}+\Gamma(\beta)}\left(\alpha+\frac{y^{\beta}}{\beta}\right)e^{-\theta y}\left[1-\frac{\left(\alpha\beta\theta^{\beta}e^{-\theta y}+\Gamma(\beta+1,\theta y)\right)}{\beta\left(\alpha\theta^{\beta}+\Gamma(\beta)\right)}\right]^{1-k}.$$

9. Maximum Likelihood Estimator of Burhan Distribution

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from Burhan distribution. Then its likelihood function is given by

$$l = \prod_{i=1}^{n} f(y_i, \alpha, \beta, \theta)$$

$$= \left(\frac{\theta^{\beta+1}}{\alpha\theta^{\beta}+\Gamma(\beta)}\right)^{n}\prod_{i=1}^{n}\left(\alpha+\frac{y_i}{\beta}\right)e^{-\theta\sum_{i=1}^{n}y_i}.$$

The log likelihood function is given as

$$log\, l = n(\beta+1)\, log\,\theta - n\, log\left(\alpha\theta^{\beta}+\Gamma(\beta)\right) + \sum_{i=1}^{n} log\left(\alpha+\frac{y_i^{\beta}}{\beta}\right) - \theta\sum_{i=1}^{n} y_i\,. \quad (9.1)$$

The partial derivatives of equation (9.1), with respect parameters are given as

$$\frac{\partial\, log\, l}{\partial\alpha} = \frac{-n\theta^{\beta}}{\alpha\theta^{\beta}+\Gamma(\beta)} + \sum_{i=1}^{n}\frac{\beta}{\alpha\beta+y_i^{\beta}} \quad (9.2)$$

$$\frac{\partial\, log\, l}{\partial\beta} = n\, log\,\theta - \frac{n\left(\alpha\theta^{\beta}\, log\,\theta+\Gamma'(\beta)\right)}{\alpha\theta^{\beta}+\Gamma(\beta)} + \sum_{i=1}^{n}\frac{(\beta\, log\, y_i - 1)y_i^{\beta}}{\beta\left(\alpha\beta+y_i^{\beta}\right)} \quad (9.3)$$

$$\frac{\partial\, log\, l}{\partial\theta} = \frac{n(\beta+1)}{\theta} - \frac{n\alpha\beta\theta^{\beta-1}}{\alpha\theta^{\beta}+\Gamma(\beta)} - \sum_{i=1}^{n} y_i. \quad (9.4)$$

From equations (9.2), (9.3) and (9.4), we have obtained a system of non-linear equations which cannot be expressed in compact form and is difficult to solve explicitly for α, β and θ. Applying the iterative methods such as Newton-Raphson method, secant method, Regula-falsi method etc. The MLE of theparameters denoted as $\hat{\zeta}\left(\hat{\alpha},\hat{\beta},\hat{\theta}\right)$ of $\zeta(\alpha,\beta,\theta)$ can be obtained by using the above methods.

For interval estimation and hypothesis tests on the model parameters, an information matrix is required. The 3 by 3 observed matrix is

$$I(\zeta) = -\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2\, log\, l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2\, log\, l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2\, log\, l}{\partial\alpha\partial\theta}\right) \\ E\left(\frac{\partial^2\, log\, l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2\, log\, l}{\partial\beta^2}\right) & E\left(\frac{\partial^2\, log\, l}{\partial\beta\partial\theta}\right) \\ E\left(\frac{\partial^2\, log\, l}{\partial\theta\partial\alpha}\right) & E\left(\frac{\partial^2\, log\, l}{\partial\theta\partial\beta}\right) & E\left(\frac{\partial^2\, log\, l}{\partial\theta^2}\right) \end{bmatrix}$$

The elements of above information matrix can obtain by differentiating equations (9.2), (9.3) and (9.4) again partially. Under standard regularity conditions when $n \to \infty$ the distribution of $\hat{\zeta}$ can be approximated by a multivariate normal $N\left(0, I\left(\hat{\zeta}\right)^{-1}\right)$ distribution to construct approximate confidence interval for the parameters.

Hence the approximate $100(1-\psi)\%$ confidence interval for α, β and θ are respectively given by

$$\hat{\alpha} \pm Z_{\frac{\psi}{2}}\sqrt{I_{\alpha\alpha}^{-1}(\hat{\zeta})}, \quad \hat{\beta} \pm Z_{\frac{\psi}{2}}\sqrt{I_{\beta\beta}^{-1}(\hat{\zeta})} \quad \text{and} \quad \hat{\theta} \pm Z_{\frac{\psi}{2}}\sqrt{I_{\theta\theta}^{-1}(\hat{\zeta})},$$

where $Z_{\frac{\psi}{2}}$ denotes the ζ^{th} percentile of the standard normal distribution.

10. Applications

This section exhibits the adaptability of the formulated distribution by applying real-world data sets. The suggested distribution is compared to the Shanker distribution (SHD), the Akash distribution (AD), the Ishita distribution (ID), the Pranav distribution (PD), the Rani distribution (RD), the Prakaamy distribution (PKD), and the Lindley distribution. We were efficient in outranking the specified distribution.

In order to compare the above distribution models, we consider the criteria like AIC (Akaike information criterion), CAIC (corrected Akaike information criterion), BIC (Bayesian information criterion). Among the above distributions, the better distribution is considered having lesser values of AIC, CAIC, HQIC and BIC.

Data set 1: This data is obtained from Murthy et al. [13], which represents failure times of 84 Aircraft Windshield. The data follows:

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632,3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281,2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278,1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166,4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Table 10.1: Descriptive statistics of data 1st.

Min	Q_1	Median	Mean	Q_3	Skew.	Kurt.	Max
0.040	1.866	2.385	2.563	3.376	0.086	2.365	4.663

Table 10.2: The ML estimates of the unknown parameters for data set 1st.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	S.E		
				$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
BHD	0.0206	5.6641	2.4893	0.0129	1.0566	0.4044
SHD	……	……	0.6440	……	…….	0.0459
AD	……	……	0.9337	……	…….	0.056
ID	……	……	0.9181	……	……	0.0495
PD	……	……	1.2104	……	……	0.0530
RD	……	……	1.5155	……	……	0.0563
PKD	……	……	1.7287	……	……	0.0710
LD	……	……	0.6296	……	……	0.0502

Table 10.3: Performance of distributions for data set 1st.

Model	$-\log l$	AIC	CAIC	BIC	HQIC
BHD	127.63	261.26	261.55	268.59	264.21
SHD	151.55	305.10	305.14	307.54	306.08
AD	147.46	296.93	296.97	299.37	297.91
ID	146.83	295.66	295.71	298.10	296.64
PD	144.40	290.81	290.85	293.25	291.79
RD	143.12	288.24	288.29	290.68	289.22
PKD	188.70	379.40	379.44	381.84	380.38
LD	153.96	309.93	309.98	312.37	310.91

Data set 2: The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [12]. The data are as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93,0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13,1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46,1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97,2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78,2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.

Table 10.4: Descriptive Statistics of data 2nd

Min	Q_1	Median	Mean	Q_3	Skew.	Kurt.	Max
0.100	1.077	1.450	1.754	2.240	1.328	4.913	5.550

Table 10.5: The ML estimates of the unknown parameters for data set 2nd.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	S.E		
				$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$
BHD	0.0067	2.4143	1.9124	0.0085	0.6517	0.3729
SHD	……	……	0.8667	……	…….	0.0675
AD	……	……	1.2227	……	…….	0.0815
ID	……	……	1.1664	……	……	0.0680
PD	……	……	1.4780	……	……	0.068
RD	……	……	1.7994	……	……	0.0699
PKD	……	……	2.1118	……	……	0.0853
LD	……	……	0.8744	……	……	0.0771

Table 10.6: Performance of distributions for data set 2nd.

Model	$-\log l$	AIC	CAIC	BIC	HQIC
BHD	94.06	194.12	194.48	200.95	196.84
SHD	105.98	213.97	214.02	214.87	216.24
AD	107.06	216.13	216.19	218.41	217.04
ID	108.03	218.06	218.12	220.34	218.97
PD	112.53	227.07	227.12	229.34	227.97
RD	118.26	238.53	238.58	240.80	239.43
PKD	176.87	355.75	355.81	358.03	356.66
LD	106.52	215.05	215.10	217.32	215.95

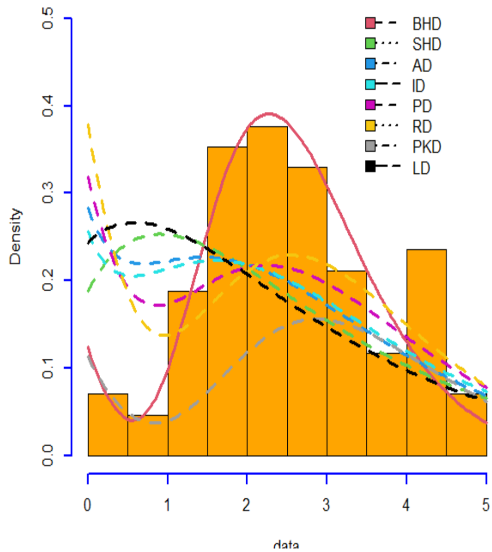

Figure 10.1: Fitted pdf's for data set I.

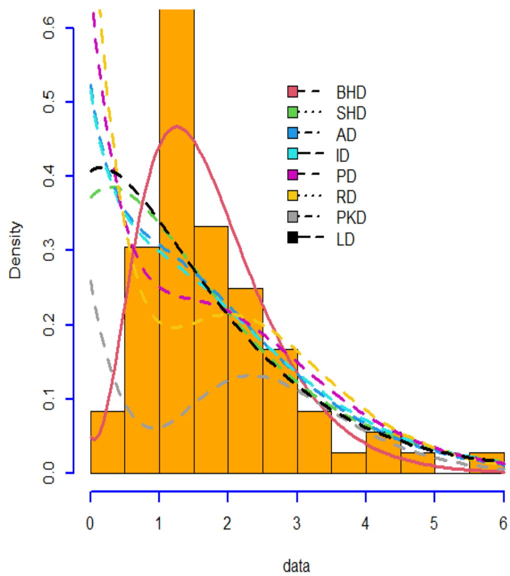

Figure 10.2: Fitted pdf's for data set II.

It is clear from Table 10.3 and 10.6 that Burhan distribution (BHD) has least values of $-log$, AIC, CAIC, BIC and HQIC when compared with competitive distributions. We accomplish that Burhan distribution provides an adequate fit than compared distributions.

11. Conclusion

In the present paper a novel distribution has been formulated by employing convex combination of exponential and gamma distribution stated as Burhan distribution (BHD). Its several properties including moments, moment generating function, survival function, hazard rate function, mean residual life function, mean deviations, order statistics, Renyi entropy and Tsallis entropy have been discussed. The parameters of the distribution have been estimated by known method of maximum likelihood estimator. Finally the performance of the model has been examined through two data sets and compared which shows that Burhan distribution gives an adequate fit for the data sets.

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